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1 Stable Marriages Problem.

We're given a complete bipartite graph consisting of n boys and n girls. Each girl (boy) makes a preference list of the boys (girls), such that the first element in the list shows the boy (girl) she (he) likes the most and the last element is the last boy (girl) in the world she (he) would like to be with. The preference lists of the girls and the boys are required to contain all the individuals from the opposite sex. A perfect matching on this graph is called a marriage and the matched pairs are called couples. We can formalize this problem as follows: given a complete bipartite graph $G = (G \cup B, E)$ and preference lists $b_i = [\sigma(b^i, 1), \sigma(b^i, 2), \ldots, \sigma(b^i), n)]$, $g_j = [\sigma(g^j, 1), \sigma(g^j, 2), \ldots, \sigma(g^j, n)]$ for all $b_i \in B, g_j \in G$, find a perfect matching that is not *unstable*.

A marriage M is called *unstable* iff the following is true: Let Jane and Paul, Mary and John be two couples in M. If Jane prefers John to Paul and John prefers Jane to Mary according to their preference lists. Then, this marriage is unstable.

Paul $\leftrightarrow$ Jane	John = [... Jane ... Mary ...]
John $\leftrightarrow$ Mary	Jane = [... John ... Paul ...]

In other words a marriage M is called unstable, iff there exists a pair that is not matched in M who prefer each other to their current partners.
The algorithm to find a stable marriage is given by Gale & Shapley and it works as follows:

Algorithm for finding a stable marriage

Step 1: Every girl proposes to the first boy in her preference list.

Step 2: Every boy who receives proposals checks his preference list and gets engaged to the girl having the highest preference among the ones who proposed to him.

Step 3: In the next round, all the girls who are not engaged propose to the next boy in their preference lists.

Step 4: If the boy is already engaged with another girl, he checks if he got a proposal from a girl with higher preference, if so (as most boys do (!)) he breaks his engagement with his former fiancée and he gets engaged to the girl who just proposed to him. In the next round, the girls who lost their fiancés in the competition, will propose to the boy that comes just after their lost fiancés in their preference lists.

Step 5: Last two steps are repeated until everybody is matched.

The above algorithm will always terminate: because, when a girl proposes to the last boy in her list all the other boys are engaged (to the other $n-1$ girls), so the boy who is not engaged has to accept her proposal.

Proposition 1.1 The marriage found by the above algorithm is *stable*.
Proof:

Assume by way of contradiction that the marriage is unstable, that is there is a pair John and Jane as explained above. Since John comes before Paul in Jane's preference list, Jane must have proposed to John before Paul. Since John rejected Jane, he must be engaged with a girl that comes before Jane in his preference list. John will break his engagement only if a girl that comes before his current partner proposes to him, therefore Mary must come before Jane in his list. Hence, the marriage is stable. $\square$

It is easy to see that the above algorithm runs in $O(n^2)$ steps.