CMSC 754: Spring 1996 Samir Khuller

Davenport Schinzel Sequences

A Davenport-Schinzel (DS) sequence is a sequence constructed over an alphabet of size n. We define a *Davenport-Schinzel* sequence of order s as follows (referred to as $DS(n,s)$). There are two conditions that these sequences are required to satisfy.

1. No two consecutive characters should be the same.
2. There should be no alternating subsequence of the form $\ldots a \ldots b \ldots a \ldots b \ldots$ of length $s+2$ (for any pair of characters a, b).

Examples: 122341 is not a DS sequence since 2 occurs consecutively. 14234132 is a DS sequence of order 3, but not of order 2, since there is an alternating subsequence 1..2..1..2.

Let $\lambda_s(n)$ be the length of a longest $DS(n,s)$ sequence. We are interested in deriving good upper bounds on $\lambda_s(n)$, as function of n. These sequences are interesting since they can be used to derive upper bounds on the complexity of lower envelopes of line segments, lower envelopes of more general functions, as well as obtaining an alternate proof of the zone theorem. They are very often used to obtain simpler proofs of the complexity of various combinatorial structures as well. In short, they are very useful! Most of the material in this lecture is taken from a recent book by Sharir and Agarwal on Davenport-Schinzel sequences. Szemeredi showed an $O(n \log^* n)$ bound for any fixed s. Subsequently, for $s = 3$ this was improved to $O(n\alpha(n))$. Tight bounds are known for other values of s as well (most of the book is devoted to the analysis of $\lambda_s(n)$).

It is easy to see that $\lambda_1(n) = n$. I figured out a fix to my proof for order 2 sequences. Let me stick my neck out once more! (The book gives a proof by induction.)

Lemma 0.1 $\lambda_2(n) \leq 2n - 1$

Proof: Consider the smallest n for which there is a $DS(n,2)$ sequence of length $2n$. Suppose there is a character that occurs only once, say a. Delete a from the sequence. If two consecutive characters become identical; we delete one of them (this might happen if we have bab as a substring). We now have a $DS(n-1,2)$ sequence of length at least $2n-2 = 2(n-1)$, a contradiction. Suppose each character occurs twice. Let the sequence be $aS'aS''$. Clearly, no character can occur both in S' and S'' (otherwise we will have an alternating pattern of length 4). Deleting both occurrences of a from the sequence results in a $DS(n-1,2)$ sequence of length $2(n-1)$, a contradiction. □

Lemma 0.2 $\lambda_3(n) \leq 2n(1 + \log n)$

Proof: We prove this by induction. Let a be the least frequently occurring character in the sequence. An upper bound on the number of times it occurs is $\frac{\lambda_3(n)}{n}$. Deleting all copies of a yields a sequence that can potentially have consecutive characters. However, at most such consecutive characters are generated (one each, from the first and last occurrence of a). Otherwise, the sequence has the form ...*a*...*bab*...*a*... and is not a DS sequence or order 3. The remaining sequence is a $DS(n-1,3)$ sequence of length $\lambda_3(n-1)$. Hence, an upper bound on $\lambda_3(n)$ is $\lambda_3(n-1) + \frac{\lambda_3(n)}{n} + 2$. Simplifying, we get $(1 - 1/n)\lambda_3(n) \leq \lambda_3(n-1) + 2$.

This is the same as $\frac{\lambda_3(n)}{n} \leq \frac{\lambda_3(n-1)}{n-1} + \frac{2}{n-1}$. Expanding the recurrence, gives the upper bound □

Zone Theorem

We now study some applications of these sequences. The first application is to obtain a new proof of the zone theorem.

Let ℓ be the new line that is being inserted. We would like to prove that the total complexity of the faces that ℓ intersects is $O(n)$. We do this as follows. Assume ℓ is horizontal. We will show that the complexity of the portion of the faces "above" ℓ is $4n$. Assume no two lines are parallel. With this assumption the

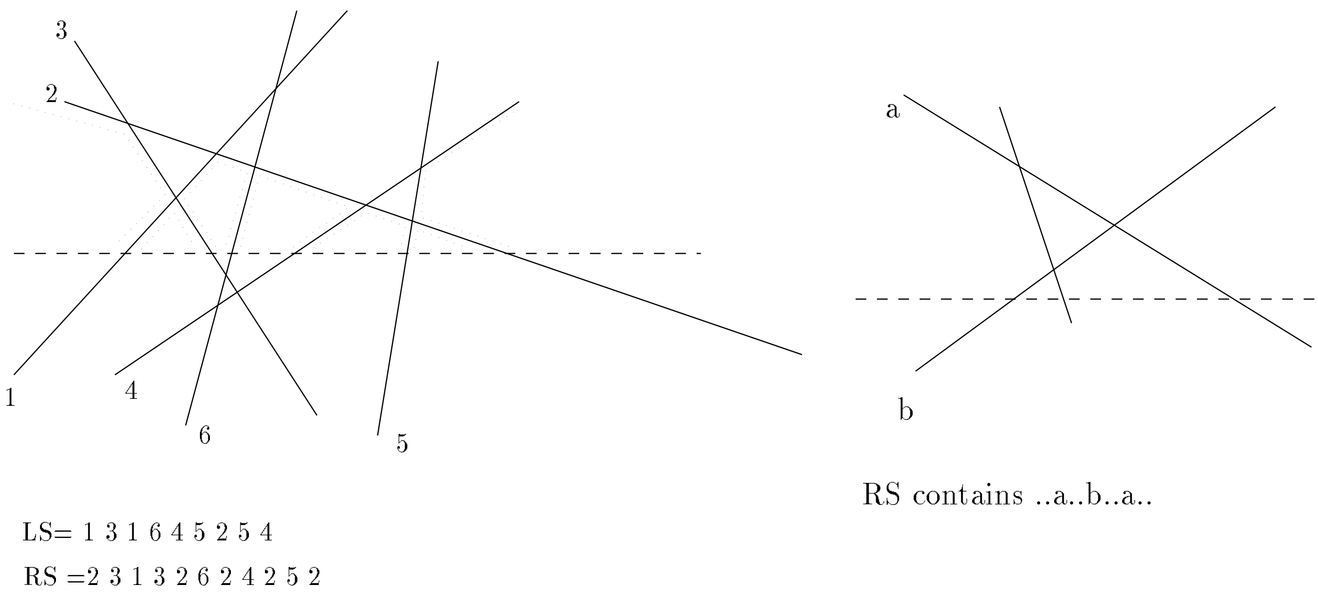

Figure 1: Proof of zone theorem.

highest vertex on any face is unique (since no other line is horizontal). As we traverse the faces in clockwise order, we output the line number each edge belongs to. Lines are labelled with distinct numbers. We create two sequences, a left edge sequence and a right edge sequence. For the edges before the highest point in the clockwise traversal of a face, the line numbers are appended to the left sequence (LS), and the edges after the highest point in the traversal, the line numbers are appended to the right sequence (RS).

We claim that each list is $DS(n, 2)$ sequence. The first condition is easy to establish. The second condition is established by a "picture-proof". Consider the right sequence. If it contains $a...b...a$ then it cannot contain b again.

This proves that the total complexity of the faces above ℓ is at most $4n$. This gives us a bound of about $8n$. Since the lines that cross ℓ were counted twice, we can reduce this to $6n$. However, the tight upper bound is $5.5n$.

Closest Pair/Nearest Neighbors of Moving Points

Let P be set of n points in the plane that are moving along predefined trajectories. Let $x_i(t)$ and $y_i(t)$ denote the position of point p_i at time t. Assume that these functions are polynomials of a fixed degree s. We fix our attention on point p_i, and let $g_{ij}(t)$ be the square of the distance from p_i to p_j at time t. Clearly, g_{ij} is a polynomial of degree $2s$. Let $G_i(t) = \min j \neq i g_{ij}(t)$. The function $G_i(t)$ is a minima of $n-1$ functions. Each pair of functions g_{ij} and g_{ik} pairwise intersect in at most $2s$ points, since that is the degree of the polynomial defining them. We now claim that the number of intersection points on the lower envelope is at most $\lambda_{2s}(n-1)$. This is because if two functions appear $2s+2$ times alternately on the lower envelope, then they must have $2s+1$ intersection points. This is an upper bound on the number of times the nearest neighbor of p_i can change. (You should make sure you understand why this is the case.) In fact, we call it a $DS(n, s)$ sequence, since it is the lower envelope of the minima of n functions that have at most s pairwise intersections.

A similar argument can be used to show that the number of times the closest pair of the n moving points can change is at most $\lambda_{2s}(C(n, 2))$.