Spring 1996 **CMSC 754: Solution to Homework 1** Samir Khuller

Due in class: February 19, 1996.

(1) At most 6 points can be placed in the "window" of size $\delta \times 2\delta$. To see this, we make an observation that we will use frequently in the proof: if two points are greater than or equal to a distance δ apart, then the two open circles of radius $\delta/2$ centered at these points do not intersect.

We now show that if all interpoint distances are greater than or equal to δ, then a maximum of three points can exist in a "window" of size $\delta \times \delta$. To see this, we assume that 4 points can exist satisfying the condition.

Now, quarter the $\delta \times \delta$–square. Note that no two points can inhabit the same quadrant because there is no way to place two open circles of radius $\delta/2$ inside a quadrant without overlap. Thus, each of the four points must live in a unique quadrant. We now claim that each of these points must be a corner point of the square, violating the x–coordinate condition. To see this, assume that at least one of the points p inhabiting a quadrant is not a cornerpoint. Clearly, then its open circle of radius $\delta/2$ must impinge upon at least one of the other quadrants. A straightforward computation (left to the reader) shows that the open circle of radius $\delta/2$ centered at any point in this neighboring quadrant must intersect the circle or radius $\delta/2$ centered at p. Thus, each of the four points must be a corner point, violating the same x-coordinate condition. Thus, a maximum of three points can exist.

The same result holds for the other square of the $\delta \times 2\delta$ "window." Thus, a maximum of six points can exist without violating the conditions. To realize these six points, normalize the coordinates so that the southwest corner is (0,0) and the northeast corner is (2,1). Then place points at: $(0,0), (1/20,1), (19/20,1/2), (21/20,1/2), (39/20,1), (2,0)$.

(2) (Essentially Yoram Sussmann's proof)

(a) Recall from the law of cosines that,

$$|v_1 - v_2|^2 = |v_1|^2 + |v_2|^2 - 2|v_1||v_2|\cos\theta$$

and observe that

$$v_1 - v_2 = (x_1 - x_2)\vec{\imath} + (y_1 - y_2)\vec{\jmath}$$

Now, rearrange as follows,

$$\begin{aligned}
2|v_1||v_2|\cos\theta &= x_1^2 + y_1^2 + x_2^2 + y_2^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 \\
&= x_1^2 + y_1^2 + x_2^2 + y_2^2 - x_1^2 - x_2^2 + 2x_1x_2 - y_1^2 - y_2^2 + 2y_1y_2 \\
&= 2x_1x_2 + 2y_1y_2
\end{aligned}$$

and see that

$$|v_1||v_2|\cos\theta = x_1x_2 + y_1y_2 = v_1 \cdot v_2$$

(b) Note that the area A of the parallelogram formed by the two vectors is $|v_1||v_2|\sin\theta$. (This can be found by recalling that the area of a parallelogram is given by the base times the height.)
Now,

$$\begin{aligned}
A^2 &= |v_1|^2|v_2|^2\sin^2\theta \\
&= |v_1|^2|v_2|^2(1 - \cos^2\theta) \\
&= |v_1|^2|v_2|^2 - |v_1|^2|v_2|^2\cos^2\theta \\
&= (x_1^2 + y_1^2)(x_2^2 + y_2)^2 - (v_1 \cdot v_2)^2 \\
&= (x_1^2 + y_1^2)(x_2^2 + y_2)^2 - (x_1x_2 + y_1y_2)^2 \\
&= x_1^2y_2^2 + y_1^2x_2^2 - 2x_1x_2y_1y_2
\end{aligned}$$

and

$$|v_1 \times v_2|^2 = (x_1y_2 - x_2y_1)^2 = x_1^2y_2^2 + x_2^2y_1^2 - 2x_1y_2x_2y_1 = A^2$$

(c) From (b) we have that $|v_1 \times v_2| = |v_1||v_2| \sin\theta$. Thus, if v_1 is clockwise from v_2 with respect to the origin, then $\sin\theta > 0$ and clearly $|v_1| > 0$ and $|v_2| > 0$. Hence, $|v_1 \times v_2|$ must be positive.

(3) The following figure gives a placement of vertex guards that can cover the boundary of the polygon, but not the interior.

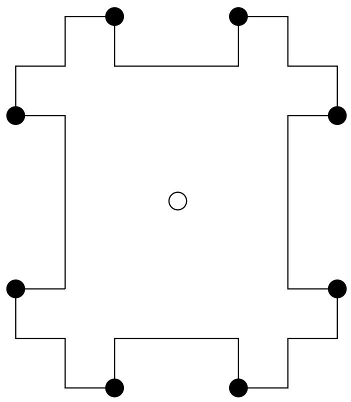

Figure 1: The boundary is covered but none of the guards can see the point denoted by the hollow circle in the center.

(4) The first point to note is that for clear visibility we need point guards. We cannot cover the boundary of a polygon by placing guards only on vertices if we are interested in clear visibility.

There were many ways to intrepret this ambigous question. One can draw a polygon where a single usual seeing vertex guard suffices, but one needs at least two clearly seeing point guards. One can also draw polygone where one clearly seeing point guard can see the entire polygon, but no vertex guard can see the entire polygon. So one cannot conclude that one type of guard is "stronger" or "weaker" than the other type.

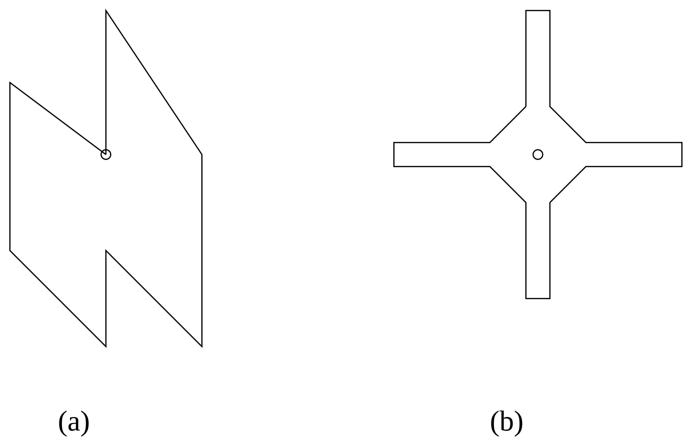

Figure 2: (a) One vertex guard suffices, but we need two clearly seeing point guards. (b) One point guard suffices, but we need two vertex guards.

There were many ways to intrepret this ambigous question; from another aspect one can argue that the worst case number of clearly seeing point guards is the same as usual vertex guards, so no extra strength is added by seeing along boundaries.

Nonetheless, Fisk's proof still holds for clearly seeing point guards; observe that any guard placed slightly off of a vertex can still see an entire triangle. Thus $G(n) = G'(n)$. To prove this, the "preturbation" of the guard's position has to be done carefully. Each guard in the proof is assigned a collection of triangles that it must see, all of which share a common vertex at the guard's location. The perturbation has to be small enough so that the guard can still see the set of triangles. An example shows why the perturbation has to be limited. The guard at vertex (a) is "covering" the set of triangles as shown. We have to be careful in making this a point guard since we do not want to lose visibility to the line segment X. Since the triangulation is proper, each triangle has a non-zero area so such a perturbation can always be done. (The "core" of a polygon is the set of points from which the entire polygon is visible – here the portion of the polygon that a single guard covers is a union of the triangles sharing a common vertex. The core of this union of triangles, is not a single point, and this is important to show that the perturbation can be done.)

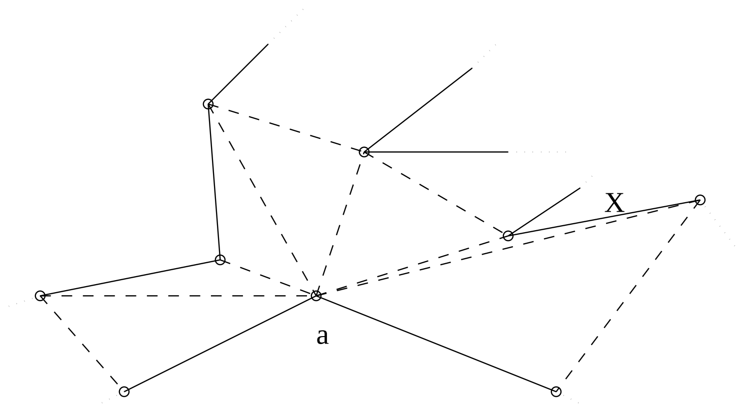

Figure 3: Perturbation has to be done carefully.