CENTERS OF SETS OF PIXELS

Samir Khuller
Department of Computer Science
University of Maryland
College Park, MD 20742-3255

Azriel Rosenfeld
Center for Automation Research
University of Maryland
College Park, MD 20742-3275

Angela Y. Wu
Department of Computer Science and
Information Systems
American University
Washington, DC, 20016-8116

Abstract

The *center* of a connected graph G is the set of nodes of G for which the maximum distance to any other node of G is as small as possible. If G is a simply connected set of lattice points ("pixels") with graph structure defined by 4-neighbor adjacency, we show that the center of G is either a 2-by-2 square block, a diagonal staircase, or a (dotted) diagonal line with no gaps.

The authors thank Fridrich Sloboda for suggesting the problem, Reinhard Klette for early discussions, and Janice Perrone for her help in preparing this paper. The first author's research is supported by NSF Career Award CCR 95-01355.

1 Introduction

The "center" of a (connected) region R is usually taken to be its centroid (or "center of gravity")—the point P that minimizes the sum of the squared distances between P and all the points of R. This is a reasonable definition for some purposes; for example, it minimizes the average (squared) travel time, "as the crow flies", from P to all the points of R. However, it has the disadvantage that P may not itself be a point of R; this can happen if R has holes (e.g., it is an annulus) or even if it is nonconvex (e.g., it is a crescent).

We can force the center to lie inside the region by redefining it in terms of "intrinsic" distance. If R is a connected region and A, B are points of R, the intrinsic distance $d_R(A, B)$ is defined as the length of the shortest path in R between A and B. We can then define the "intrinsic centroid" of R as the point P of R that minimizes $\sum d_R^2(P, Q)$, summed over all the points Q of R. This minimizes the average (squared) travel time, by a "ground vehicle" that must stay inside R, from P to all the points of R.

Whether we use ordinary ("extrinsic") or intrinsic distance, we can also define centers that minimize quantities other than the sum of squared distances. For example, we can choose the P that minimizes the maximum (extrinsic or intrinsic) distance between P and all the points of R; we might call such a P the "min-max (intrinsic) center". It is known [1] that the min-max intrinsic center (also called the "geodesic center") of a simple polygon is a unique point.

Any of these definitions is applicable if R is any (pathwise) connected subset of a metric space. In particular, the space can be discrete—for example, a graph or a digital image. When the space is discrete, the center may not be unique—in other words, exact ties may occur. In fact, the center can even be the entire space; for example, this is true for graphs such as a cycle or a clique. For an acyclic graph (i.e., a tree), however, it can be shown [2] that the min-max center is either a single node or two adjacent nodes. Min-max centers of graphs, as well as various other types of "centers", have been studied by many researchers; for a recent review see [3].

In this paper we characterize min-max intrinsic centers for an important class of discrete spaces: the lattice points in the plane under the graph structure defined by 4-neighbor adjacency. As we shall see, in this space, the min-max intrinsic center of a simply connected set of lattice points is either a 2-by-2 square block, a diagonal staircase, or a dotted diagonal line with no gaps (see Fig. 1); note that in the latter two cases, the center can be arbitrarily large.

Sets of lattice points ("pixels") have been extensively studied in *digital geometry*; for an introduction to this subject see [4]. Such sets arise when planar regions are digitized; they can be regarded as discrete approximations of these regions. But as our results show, the center of a digital region may not be a very good approximation to the (intrinsic min-max) center of the original planar region, since it can be an arbitrarily long staircase or dotted diagonal.

Section 2 of this paper reviews the concepts of digital geometry that we will use. Section 3 characterizes the min-max intrinsic centers of simply connected sets of pixels under 4-neighbor adjacency. It would be of interest to extend our results to other types of lattice-point adjacencies in two or three dimensions; in Section 4 we discuss the case of 8-neighbor

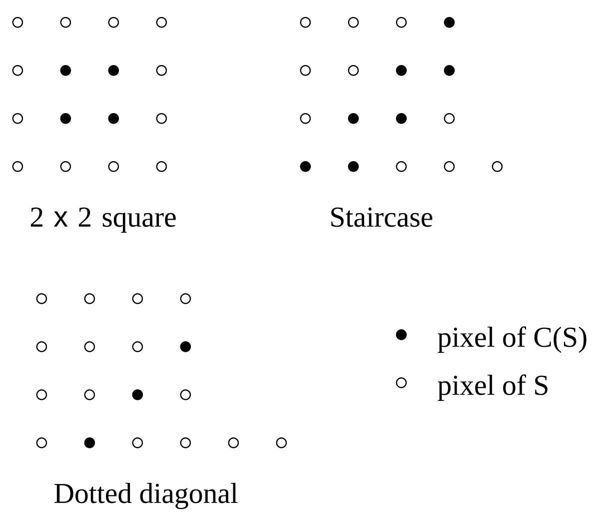

Figure 1: The possible types of centers

adjacency in the plane.

2 Sets of pixels: Connectedness and distance

The *lattice points* in the plane, i.e. the points whose coordinates are integers, will be called *pixels* (short for "picture elements"). Any pixel $a = (i, j)$ has four horizontal and vertical neighbors $(i \pm 1, j)$, $(i, j \pm 1)$. These neighbors are called the (4-)*neighbors* of a. We will sometimes refer to them as the north, east, south, and west neighbors of a. Neighbors are also said to be *adjacent*.

A *path* p from a to b is a sequence of pixels $a = a_0, a_1, \ldots, a_n = b$ $(n \geq 0)$ such that a_i is adjacent to a_{i-1}, $1 \leq i \leq n$; here n is called the *length* of p. A set S of pixels is called *connected* if for any a, b in S there exists a path $a = a_0, \ldots, a_n = b$ such that all the a_i's belong to S (in brief: a path in S from a to b). Evidently the reversal p^{-1} of a path is a path, and a concatenation of paths is a path. A path whose endpoints are the same $(a_0 = a_n)$ is called a *cycle*.

Analogous concepts of adjacency, paths, connectedness, etc. can be defined if we redefine neighbor to include the diagonal neighbors $(i \pm 1, j \pm 1)$ of (i, j). A finite connected set of pixels S is called *simply connected* if its complement $\bar{S}$ is connected in the 8-neighbor sense [4]. [In general, $\bar{S}$ can be partitioned into a finite number of maximal 8-connected subsets, called its 8-connected *components*. Exactly one of these components is infinite; it is called the *background* of S. The other components, if any, are called *holes* in S.]

The *city block distance* between two pixels $a = (i, j)$ and $b = (h, k)$ is $|i - h| + |j - k|$. Evidently the pixels at city block distance 1 from a are just the (4-)neighbors of a.

Let S be a finite connected set of pixels. The *intrinsic distance* d_S between two pixels a, b of S is the length of the shortest path in S from a to b. Evidently, $d_S(a, b)$ is at least equal to the city block distance between a and b.

The *eccentricity* of any pixel a in S is the greatest intrinsic distance from a to any pixel of S. The minimum of the eccentricities of the pixels of S is called the *radius* of S. The *center* of S, which we denote by $C(S)$, is the set of pixels of S with minimal eccentricity. In the next section we characterize the centers of simply connected sets of pixels.

3 Centers of simply connected sets of pixels

Let a, b be two pixels of S; without loss of generality, let $a = (i, j), b = (h, k)$, where $i \leq h, j \leq k$, so that b is northeast of a. Let p be a shortest path in S from a to b; thus p consists of an alternation of horizontal and vertical runs of two or more pixels. If p consists entirely of northward and eastward runs, its length is $(h-i)+(k-j)$, which is the city block distance from a to b. Suppose p involves runs in a third direction, say southward; then there exists a horizontal run r that is preceded by a northward run and succeeded by a southward run (or vice versa). Since p is a shortest path, some pixel c of r must have a pixel of $\bar{S}$ as its south (or north) neighbor; if not, r could be replaced by the horizontal run consisting of the south (north) neighbors of its pixels, so that p could be shortened. Note that in this situation we have $d_S(a, b) > (h-i) + (k-j)$.

Proposition 1 *Let p be a shortest path in S, and let C be a horizontal or vertical run of pixels of S. Then p can intersect C in at most one run of pixels.*

Proof: If p intersects C in the nonadjacent runs r, s, the subpath of p between r and s is not a straight line segment. Hence p could be shortened by replacing this subpath by the segment of C between r and s; but this is impossible since p is a shortest path in S. □

Let p be a shortest path in S, let C be a horizontal or vertical run of pixels of S, and let p intersect C in the run r. If r is a single pixel (not an endpoint of p), or if r is a run of p (not the first or last run) and the runs of p preceding and following p are in the same direction (e.g., r is a vertical run, and the preceding and following runs are both eastward or both westward), we say that p *crosses* C at r. In the following proposition, C is vertical, but the analogous result evidently holds if C is horizontal.

Proposition 2 *Let S be simply connected, let C be a maximal vertical run of pixels of S, and let p be a shortest path in S, say from a to b, that crosses C. Then a and b are in different components of $S - C$.*

Proof: Let $\hat{S}$ be the union of the unit squares centered at the pixels of S. Since S is simply connected, the border B of $\hat{S}$ is a simple closed curve. (B may touch itself at corners of squares, but the border following algorithm [4] unambiguously determines which sides of the edges of the squares are inside $\hat{S}$ and which are outside.) Let L be the line segment joining the pixels of C, extended until it meets B (at the centers of the top and bottom edges of the unit squares centered at the top and bottom pixels of C). The intersection points of L with B divide B into two arcs B_1, B_2, and the concatenations of L with B_1 and B_2 are simple closed curves K_1 and K_2. Evidently the only pixels of S that lie on K_1 or K_2 are those that lie on L; all other pixels of S are inside K_1 or inside K_2 (but not both). Thus any component of $S - C$ is either inside K_1 or inside K_2 (but not both). Since p crosses C (say at r), the pixel preceding r on p must be inside K_1 and the pixel following r on p must be outside it (or vice versa), and the reverse is true for K_2. Since p can only intersect C in a single run, all the pixels preceding r on p must be inside K_1 and all the pixels following r on p must be inside K_2 (or vice versa); hence a and b are in different components of $S - C$. $\square$

Proposition 3 *Let S, C, p be as in Proposition 2, and let c be any pixel of C. If either of the following conditions holds then a and b cannot both be in $C(S)$.*

1. *For every $c' \in C$, $d_S(c, c') < \min(d_S(c', a), d_S(c', b))$.*
2. *$c \notin C(S)$, and for every $c' \in C$, $d_S(c, c') \leq \min(d_S(c', a), d_S(c', b))$.*

Proof: Let S have radius ρ and let $f(c)$ be a pixel of S farthest from c. Evidently $d_S(c, f(c) \geq \rho$, and $> \rho$ if $c \notin C(S)$. We first observe that if a and b are both in $C(S)$, then condition (1) or (2) implies that $f(c)$ cannot be in C. Indeed, if (1) holds we have $d_S(c, f(c)) <$ min $(d_S(c', a), d_S(c', b)) \leq \rho$, contradiction; and if (2) holds we have $d_S(c, f(c)) \leq$ min $(d_S(c', a), d_S(c', b)) \leq \rho$, contradiction.

Since by Propositon 2, a and b are in different components of $S - C$, they cannot both be in the same component as $f(c)$; let a be in a different component. Thus a shortest path q from a to $f(c)$ intersects C, say at c'. If (1) holds, c' is closer to c than to a; hence if we replace the subpath of q from a to c' by the line segment cc', we obtain a path from c to $f(c)$ strictly shorter than ρ, a contradiction. Similarly, if (2) holds, and we replace the subpath of q from a to c' by the line segment cc', we obtain a path from c to $f(c)$ of length at most ρ, contradicting the fact that $c \notin C(S)$. $\square$

Proposition 4 *Let S be simply connected, and let $a = (i, j), b = (h, k)$ be two pixels of S such that $d_S(a, b) > (h - i) + (k - j)$; then a and b cannot both be in $C(S)$.*

Proof: Let p be a shortest path in S from a to b, and let c be a pixel of p (as described in the first paragraph of this section) whose south neighbor is in $\bar{S}$. [The proof for other types of c's is analogous.] Let C be the maximal vertical run of pixels of S that contains c; thus C extends northward from c. We will prove that every pixel of C is closer to c than it is to either a or b; our conclusion then follows from Proposition 3 (condition (1)).

Evidently p crosses C at c, and so cannot contain any other pixel of C. If a and b are on or below the row R containing c, any pixel c' of C is clearly closer to c than to a or b. If a is above R, and a shortest path q from a to c' in C goes as low as R, c' is clearly closer to c than to a; thus we can assume that for every c' in C, some such q stays above r (and similarly for b). Let p' be the subpath of p between its last intersection with q and its run r on R, and let q' be the subpath of q between that intersection and c'. Thus p' begins above R and ends by reaching R from below, in a northward run; hence it must have a southward run just preceding a northward run (with a horizontal run r' between them that lies below R), and some pixel of r' must have a pixel z of $\bar{S}$ as its north neighbor, since otherwise p' could be shortened. Evidently, concatenation of q', $c'c$, the segment of r up to c (reversed), and p' (reversed) yields a simple closed curve, and z must be inside this curve, contradicting the simple connectedness of S. $\square$

Proposition 5 *Let S be simply connected, and let a and b be two non-adjacent pixels in the same row (or column); then a and b cannot both be in $C(S)$.*

Proof: We give the proof for a and b in the same row; the other case is exactly analogous. If $d_S(a,b) > h-i$, a and b cannot both be in $C(S)$ by Proposition 4. If $d_S(a,b) = h-i$, all the pixels on the row between a and b are in S. Let c be any pixel strictly between a and b on that row, and C be the maximal vertical run of pixels of S that contains c. Since condition (1) of Proposition 3 evidently holds, a and b cannot both be in $C(S)$. □

Theorem 6 *A connected component of $C(S)$ is either a 2×2 square or a diagonal staircase.*

Proof: Let D be the component, and let a and b be two adjacent pixels of D. Without loss of generality assume that b is the east neighbor of a. If b has another neighbor c in D, by Proposition 5 c must be either the north or south neighbor of b. Suppose it is the north neighbor (a similar argument holds for the south case). If c has another neighbor d in D, there are two possibilities:

1) d is the west neighbor of c. In this case a, b, c, d form a 2-by-2 block of pixels, and there can be no other pixels in D since otherwise D would contain three consecutive pixels in a row or column.

2) d is the east neighbor of c. The only other neighbor of d that can be in D is its north neighbor; otherwise D would contain three pixels in a row. This north neighbor can only have an east neighbor in D; and so on. This process results in a diagonal staircase of arbitrary length. □

Theorem 7 *If $C(S)$ has more than one connected component, then each connected component is a singleton and they lie on a single diagonal with no gaps.*

Proof: Suppose a and b are in different connected components of $C(S)$. Of all such pairs, pick a pair at minimal distance. For convenience choose a coordinate system in which a is at the origin.

By Proposition 4 the intrinsic distance from $a = (0,0)$ to $b = (h,k)$ must be $h+k$. Let $0 < k \leq h$, i.e., let b be to the right of a and above a, and let the line segment ab make an angle of at most 45^o with the x-axis. (The treatment of the other relative positions of a and b is analogous.) Since a and b are not adjacent, by Proposition 5 they cannot be on the same row; hence $k > 0$.

We will now show that h must be 1; otherwise, using the following lemma together with condition (2) of Proposition 3, we can derive a contradiction.

Lemma 8 *If $h > 1$ then there exists a shortest path p in S from a to b, and a pixel $c \notin C(S)$ on p, such that for every pixel c' of C we have $d_S(c',c) \leq \min(d_S(c',a), d_S(c',b))$, where C is a maximal vertical or horizontal run of pixels of S that contains c.*

Proof: A shortest path p from a to b begins by going east or north from a. Suppose first that it goes east. If $(1,1) \in S$ and there is a shortest path from a to b through $(1,1)$ then choose $c = (1,1)$ and let C be the maximal vertical run of pixels of S through c. Note that $c \notin C(S)$ since c is closer than b to a. (The same is true in the other cases considered below.) For any c' on C, if c' is below the x-axis, its distance to c is at most its distance to a or b. If c' is above the x-axis, it is closer to c than to a. The set of pixels at the same distance as c from c' is the northeast diagonal emanating from c. Since b is on or below this diagonal, $d_S(c', c) \leq d_S(c', b)$. If there is no shortest path from a to b through $(1,1)$, let p first have y-coordinate 1 at position $(x,1)$ (some pixel on p must have y-coordinate 1 since $k > 0$). Between $(1,1)$ and $(x,1)$ there must be a pixel $(z,1)$ in $\overline{S}$; otherwise we could find a shortest path p from a to b through $(1,1)$. Then the pixels $(1,0), \ldots, (z,0)$ must all be on p, hence in S, since p moves east from $(0,1)$ and does not reach the row above $(0,0)$ until $(x,1)$. Let c be $(z,0)$, and let C be the maximal vertical run of pixels of S through c. Thus C extends downward from c, so that for any c' on C its distance to c is less than its distance to a or b.

Similarly, if $(1,1) \notin S$, let $c = (1,0)$ and let C be the maximal vertical run of pixels of S through c. Here too, C extends downward from c, so that the distance from any c' on C to c is less than its distance to a or b.

Next, suppose that a shortest path begins by going north from a. If $(1,1) \in S$ and there is a shortest path from a to b through $(1,1)$, choose $c = (1,1)$ and let C be the maximal vertical run of pixels of S through c. For any c' on C, if c' is above the x-axis, it is closer to c than to a. The set of pixels at the same distance as c from c' is the northeast diagonal emanating from c. Since b is on or below this diagonal, $d_S(c', c) \leq d_S(c', b)$. If c' is on or below the x-axis, its distance to a and c is the same, and its distance to b is greater. If there is no shortest path from a to b through $(1,1)$, let p have x-coordinate 1 at position $(1,y)$ (some pixel on p must have x-coordinate 1 since $h > 1$). Between $(1,1)$ and $(1,y)$ there must be a pixel $(1,z)$ in $\overline{S}$; otherwise we could find a shortest path p from a to b through $(1,1)$. Let c be $(0,z)$, and let C be a maximal horizontal run of pixels of S through c. Thus C extends leftward from c, so that for any c' on C its distance to c is less than its distance to a or b.

Similarly, if $(1,1) \notin S$, let $c = (0,1)$ and let C be the maximal horizontal run of pixels of S through c. Here too, C extends leftward from c, so that the distance from any c' on C to c is less than its distance to a or b. □

Lemma 8 and condition (2) of Proposition 3 lead to a contradiction. We conclude that $h = 1$, which implies that $k = 1$, so that a and b are diagonal neighbors. We now argue that the connected components of $C(S)$ are singletons!

Let a and b be in positions $(0,0)$ and $(1,1)$. Let x be a pixel that belongs to a's component and is adjacent to a. The possible positions of x are $(-1,0)$ or $(0,-1)$. We deal with the former case, since the latter case is analogous. Suppose $(0,1) \notin S$. Then $(1,0) \in S$ since $d_S(a,b) = 2$. The shortest path p from x to b goes through a. Let C be the maximal vertical run of pixels of S through a; since C extends below a, every pixel on C is closer to a than to x or b. Hence by condition (1) of Proposition 3 (with $c = a$), x and b cannot both be in $C(S)$. Suppose next that $(0,1) \in S$. Let $c = (0,1)$; then $c \notin C(S)$ since a and b are not in the same component of $C(S)$. Let C be the maximal vertical run of pixels of S through c. The distance from any pixel on C to c is at most its distance to either x or b. Hence by condition (2) of Proposition 3 (with $a = x$), x and b cannot both be in $C(S)$.

We have thus shown that the components of $C(S)$ are singletons, and that any two of them are diagonal neighbors. Suppose b is the northeast neighbor of a, and c is another component that is a diagonal neighbor of b. Then c must be northeast of b, since if it were northwest or southeast it would be in the same column or row as a, contradicting Proposition 5. Hence $C(S)$ lies on a single diagonal. $\square$

4 Centers of simply 8-connected set of pixels

In Section 3 we characterized the centers of simply connected sets of pixels under 4-neighbor adjacency. In this section we discuss the centers of simply connected sets using 8-neighbor adjacency.

The 8-neighbors of a pixel $a = (i, j)$ are its four horizontal and vertical neighbors $(i \pm 1, j), (i, j \pm 1)$ together with its four diagonal neighbors $(i \pm 1, j \pm 1)$. An 8-path p from a to b is a sequence of pixels $a = a_0, a_1, \ldots, a_n = b$ $(n \geq 0)$ such that successive pixels are 8-neighbors; here n is called the length of p. A set S of pixels is 8-connected if for any two pixels a, b in S there exists a path from a to b such that all the pixels on the path are in S. A finite 8-connected set S of pixels is called simply 8-connected if its complement $\bar{S}$ is 4-connected.

The *chessboard distance* between two pixels $a = (i, j)$ and $b = (h, k)$ is $\max(|i - h|, |j - k|)$. The intrinsic 8-distance, the eccentricity and the center of a set of pixels are defined analogously to those defined using city block distance and 4-adjacency.

As in the 4-connected case, the center of a simply 8-connected set of pixels can contain any number of pixels. Some examples of centers of simply 8-connected sets are shown in Figure 2. These examples suggest that, analogously to the 4-connected case, the pixels in the center of a simply 8-connected set are on two vertical or two horizontal straight lines. However, because shortest paths can include diagonal moves, some of the propositions we established in the 4-connected case no longer hold for 8-connected sets of pixels. For example, Proposition 1 is not true: a shortest 8-path p between two pixels of a simply 8-connected set S may intersect a horizontal or vertical run C of pixels in more than one run of pixels. A shortest 8-path p intersects a diagonal run D of pixels in at most one run, but $S - D$ remains 8-connected even if D is a maximal diagonal run of pixels.

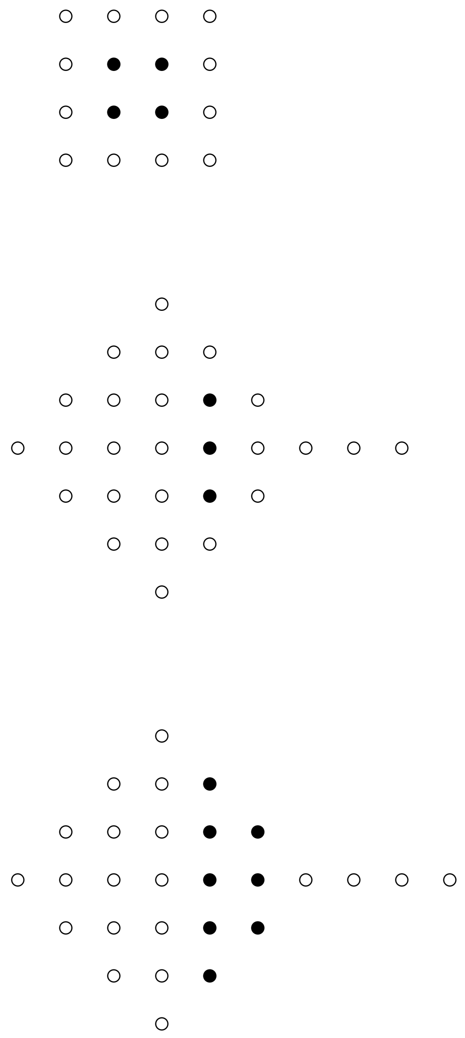

Figure 2: Possible centers in the 8-neighbor case

5 Concluding Remarks

When we use city block distance and 4-neighbor adjacency, the center of a simply connected set of pixels is either a 2×2 block, a (dotted) diagonal line segment with no gaps, or a diagonal staircase (two adjacent diagonal line segments). When we use chessboard distance and 8-neighbor adjacency, our examples show that the center of a simply 8-connected set of pixels can contain arbitrarily many pixels, but they all lie on at most two horizontal or two vertical lines. It would be of interest to characterize the centers of 8-connected sets of pixels, and the centers of lattice points in other types of grids, in both two and three dimensions.

6 References

1. R. Pollack, M. Sharir, and G. Rote, Computing the geodesic center of a simple polygon, *Discrete and Computational Geometry* **4**, 1989, 611–626.

2. C. Jordan, Sur les assemblages de lignes, *J. reine angewandte Mathematik* **70**, 1869, 185–190.

3. F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, CA, 1990, Ch. 2.

4. A. Rosenfeld, *Picture Languages*, Academic Press, New York, 1979, Ch. 2.