Bandits, Experts, and Games

CMSC 858G Fall 2016
University of Maryland

Intro to Probability*

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* Many of the slides adopted from Ron Jin and Mohammad Hajiaghayi

Outline

- Basics: “discrete” probability
- Basics: “continuous” probability
- Concentration inequalities

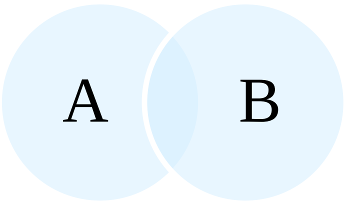
Random events

- ***Experiment***: e.g.: toss a coin twice
- ***Sample space***: possible outcomes of an experiment
  - S = {HH, HT, TH, TT}
- ***Event***: a subset of possible outcomes
  - A={HH}, B={HT, TH}
  - complement $\bar{A}$={HT, TH, TT}
  - disjoint (mutually exclusive) events: if $A \cap B = \emptyset$.
- Shorthand:
  - AB for $A \cap B$
- For now: ***assume finite #outcomes***

Definition of Probability

- ***Probability of an outcome u:***
  a number assigned to u, $\Pr(u) \geq 0$
  - Two coin tosses: {HH, HT, TH, TT}
    each outcome has probability ¼.
  - Axiom: $\sum_{u \in S} \Pr(u) = 1$
- ***Probability of an event A ⊂ S***:
  a number assigned to event: $\Pr(A) = \sum_{u \in A} \Pr(u)$
- ***Probability space***:
  - sample space S
  - probability $\Pr(u)$ for each outcome $u \in S$

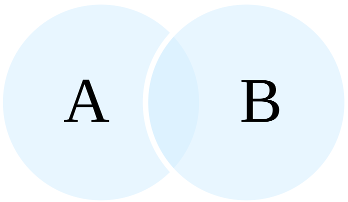
Joint Probability



- For events A and B,
  joint probability Pr(AB) (also written as Pr(A ∩ B))
  is the probability that both events happen.
- Example: A={HH}, B={HT, TH},
  what is the joint probability Pr(AB)?

  Zero

Independence



- Two events ***A and B are independent*** if

$$\Pr(AB) = \Pr(A)\Pr(B)$$

"Occurrence of *A* does not affect the probability of *B*"

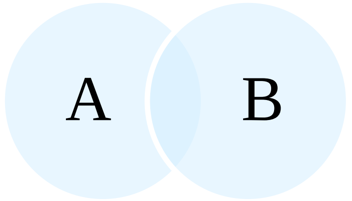
- **Prop:** $\Pr(\bar{A}B)=\Pr(\bar{A})\Pr(B)$
- Proof: $\Pr(AB)+\Pr(\bar{A}B)=\Pr(B)$

$$\begin{aligned}\Pr(\bar{A}B) &= \Pr(B)-\Pr(AB) \\ &= \Pr(B)-\Pr(A)\Pr(B) \\ &= \Pr(B)(1-\Pr(A)) = \Pr(B)\Pr(\bar{A}).\end{aligned}$$

- Events $\{A_i\}$ are ***mutually independent*** in case

$$\Pr\left(\bigcap_i A_i\right) = \prod_i \Pr(A_i)$$

Independence: examples

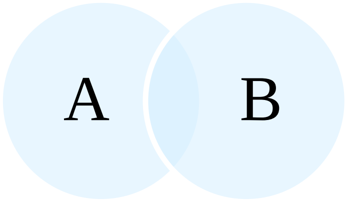


- Recall A and B are independent if Pr(AB) = Pr(A)Pr(B)
- Example: Medical trial
  4000 patients

	Women	Men
Success	200	1800
Failure	1800	200

  - choose one patient unif. at random: each patient chosen w/prob 1/4000
  - A = {the patient is a Woman}
    B = {drug fails}
  - Is event A be independent from event B ?
  - Pr(A)=0.5, Pr(B)=0.5, Pr(AB)=9/20

Independence: examples



- Consider the experiment of tossing a coin twice
- Examples: is event A independent from event B?
  - A = {HT, HH}={Coin1=H}, B = {HT}
  - A = {HT}, B = {TH}
- Disjoint ≠ Independence
- If A is independent from B, B is independent from C, is A independent from C?

Not necessarily, say A=C

Conditional probability

- If A and B are events with Pr(A) > 0, ***conditional probability of B given A*** is

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)}$$

- Example: medical trial

	Women	Men
Success	200	1800
Failure	1800	200

Choose one patient at random
A = {Patient is a Woman}
B = {Drug fails}
Pr(B|A) = 18/20
Pr(A|B) = 18/20

- If A is independent from B, Pr(A|B)= P(A)

Conditional Independence

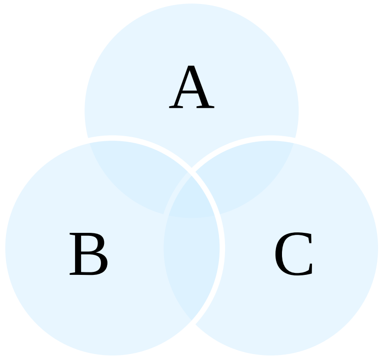
- Event A and B are ***conditionally independent given C*** if

  $$\Pr(AB|C) = \Pr(A|C)\Pr(B|C)$$

- Events $\{A_i\}$ are conditionally mutually independent given C if

  $$\Pr(\cap_i A_i|C) = \Pi_i \Pr(A_i|C)$$

Conditional Independence (cont'd)



- Example: three events A, B, C
  - $\Pr(A) = \Pr(B) = \Pr(C) = 1/5$
    $\Pr(AC) = \Pr(BC) = 1/25$, $\Pr(AB) = 1/10$
    $\Pr(ABC) = 1/125$
  - Are A, B independent? $1/5*1/5 \neq 1/10$
  - Are A, B conditionally independent given C?

    $\Pr(A|C)= (1/25)/(1/5)=1/5$,
    $\Pr(B|C)= (1/25)/(1/5)=1/5$

    $\Pr(AB|C)=(1/125)/(1/5)= 1/25= \Pr(A|C)\Pr(B|C)$
- A and B are independent
  $\neq$ A and B are conditionally independent

Random Variable

- ***Experiment***: e.g.: toss a coin twice
  - sample space S and probability $\Pr(\cdot)$
- A ***random variable*** X assigns a number to every outcome
  - X = #heads
  - "function from sample space to numbers"
  - shorthand: RV for "random variable"
- ***Distribution*** of X assigns probability $\Pr(X = x)$ to every $x \in \Re$
  - ***probability mass function*** (pmf) $f_X(x) = \Pr(X = x)$
- ***Support*** of X is the set of all $x \in \Re$ for which $f_X(x) > 0$

Random Variable: Example

- Experiment: three rolls of a die.
  Let X be the sum of #dots on the three rolls.
- What are the possible values for X?
- Pr(X = 3) = 1/6*1/6*1/6=1/216,
- Pr(X = 5) = ?

Expectation

- Expectation of random variable X

$$E[X] = \sum_x x \Pr(X = x)$$

  - weighted average of numbers in the support
- Nice properties:
  - $E[c] = c$ for any constant c.
  - Additive: $E[X + Y] = E[X] + E[Y]$
  - Linear: $E[\alpha X] = \alpha E[X]$ for any $\alpha \in \Re$
  - Monotone: if $X \leq Y$ with prob. 1, then $E[X] \leq E[Y]$

Conditional expectation

- ***Conditional expectation*** of RV X given event A:

$$E[X|A] = \sum_{x \in \text{suport}} x \Pr(X = x|A)$$

  - same formula as $E[X]$, but with conditional probabilities
- = expectation of X in a "conditional" probability space
  - same sample space as before
  - all probabilities conditioned on A
- same nice properties as before

Variance

- ***Variance*** of RV X:

$$Var(X) = E\left((X - E[X])^2\right) = E(X^2) - (E[X])^2$$

  - characterizes how much X spreads away from its expectation
- Nice properties:
  - $Var(X) \geq 0$
  - $Var(X + c) = Var(X)$ for any constant c
  - $Var(\alpha\, X) = \alpha^2\, Var(X)$ for any $\alpha \in \Re$
- ***standard deviation*** $\sigma(X) = \sqrt{Var(X)}$
- NB: variance can be infinite!
  - $X = 2^i$ with probability 2^{-i}, for each $i = 1,2,3, \ldots$.

Uniform distribution

- choose “uniformly at random” (u.a.r.)
  - sample space: K items
  - same probability $\frac{1}{K}$ for each item.
- (discrete) uniform distribution
  - random variable X can take K possible values
  - all values have the same probability $\frac{1}{K}$

Bernoulli & Binomial

- ***Bernoulli*** distribution
  - success with probability p, failure otherwise
  - ***Bernoulli*** RV X (a.k.a. 0-1 RV):
    $$\Pr(X = 1) = p \text{ and } \Pr(X = 0) = 1 - p$$
  - E[X] = p, Var(X) = E[X^2]-E[X]^2 =p-p^2
- ***Binomial distribution***
  - X = #successes in n draws of a Bernoulli distribution
  - X_i~Bernoulli(p), $i = 1 \dots n$
    $X = \sum_{i=1}^{n} X_i$, X~Bin(p, n)
  - E[X] = np, Var(X) = np(1-p)

Independent RVs

- Two random variables X and Y on the same experiment
  - outcomes of two coin tosses
- Joint distribution: $f_{X,Y}(x, y) = \Pr(X = x, Y = y)$
- X and Y are ***independent*** if for all $x, y \in \Re$

$$f_{X,Y}(x, y) = \Pr(X = x) \Pr(Y = y)$$

  - equiv.: if events $\{X = x\}$ and $\{Y = y\}$ are independent
- Basic properties:

$$E[XY] = E[X]\, E[Y]$$

$$\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)$$

- RVs $X, Y, Z, \ldots$ ***mutually independent*** if

$$\Pr(X = x, Y = y, Z = z, \ldots) = \Pr(X = x) \Pr(Y = y) \Pr(Z = z) \ \ldots$$

- Shorthand: ***IID*** for "independent and identically distributed"

Outline

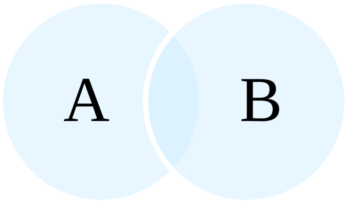
- Basics: “discrete” probability
- Basics: “continuous” probability
- Concentration inequalities

Infinitely many outcomes

- experiments can have infinitely many outcomes
  - all finite sequences of coin tosses
  - *countably* many outcomes => same treatment as before
- experiments can have ***"continuously"*** many outcomes
  - throw a dart randomly into a unit interval
    Outcomes: all numbers in [0,1]
  - infinite sequence of coin tosses
    Outcomes: infinite binary sequences
- Sample space S: set of all possible outcomes
  - Events: subsets of S
- Probabilities assigned to events, not to individual outcomes!

Definition of Probability

- ***Probability of an event*** : a number assigned to event Pr(A)
  - Axiom 1: $0 <= \Pr(A) <= 1$
  - Axiom 2: $\Pr(S) = 1$, $\Pr(\emptyset) = 0$
  - Axiom 3: For any two events A and B,
    $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(AB)$



- Corollaries
  - $\Pr(\bar{\boldsymbol{A}}) = 1 - \Pr(A)$
  - For every sequence of disjoint events $\Pr(\bigcup_i A_i) = \sum_i \Pr(A_i)$

Probability space

- ***Probability space*** consists of three things:
  - sample space S
  - set of events $\mathcal{F}$ (where each event is s subset of S)
  - probability $\Pr(A)$ for each event $A \in \mathcal{F}$
- $\mathcal{F}$ is the set of events that "we care about"
  - OK to care about some, but not all events
    ($\mathcal{F}$ does not have to include all events)
  - $\mathcal{F}$ must satisfy some formal properties ("σ-algebra")
    to make probability well-defined

Random variable X

- ***Experiment***: infinite sequence of coin tosses
  - sample space: infinite binary sequences $(b_1, b_2, \ldots)$
- A ***random variable*** X assigns a number to every outcome
  - $X = 0.b_2 b_4 b_6 \ \ldots \in [0,1]$
  - "function from sample space to numbers"
- ***Distribution*** of X: assigns probability to every interval:
$$\Pr(a \leq X \leq b)$$
  - ***cumulative distribution function*** (cdf)
$$F_X(x) = \Pr(X \leq x)$$

Continuous vs discrete

- ***"Continuous"*** random variable X:
  - each possible value happens with zero probability
  - "throw a dart randomly into a unit interval"
- ***"Discrete"*** random variable Y:
  - each possible value happens with positive probability
  - #heads in two coin tosses
  - NB: may happen even if #outcomes is infinite, e.g.:

$$\Pr(Y = i) = 2^{-i}, \qquad i = 1,2,3, \ldots$$

- RVs can be neither "continuous" nor "discrete"! E.g., max(X,Y)

Probability density function (pdf)

- ***Pdf*** for random variable X is a function $f_X(x)$ such that

$$\Pr(a \leq X \leq b) = \int_a^b f_X(x)\, dx$$

  - not guaranteed to exist (but exists in many useful cases)
- ***Support*** of X = {all x such that $f_X(x) > 0$}
  - How to define "support" if pdf does not exist? E.g.:
    - Y is discrete random variable, and
      $Z = X$ with probability ½ , and $Z = Y$ otherwise.
    - Then support(Z) = support(X) ∪ support(Y)

Expectation

- If pdf f_X exists, then expectation is

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

- General definition (for any random variable)
  - Lebesgue integral of X with respect to measure $\Pr(\cdot)$
  - no need to know what it is, for this course
- Same nice properties as in the discrete case

Uniform distribution

- Informally:
  - " Throw a random dart into an interval $[a, b]$ "
  - " each number has the same probability "
- Formally:
  - sample space: all numbers in $[a, b]$
  - probability density function: $f_X(x) = 1/(b - a)$
  - equivalently:

$$\Pr(a' \leq X \leq b') = (b' - a')/(b - a)$$

for every interval $[a', b'] \subset [a, b]$

Independent RVs

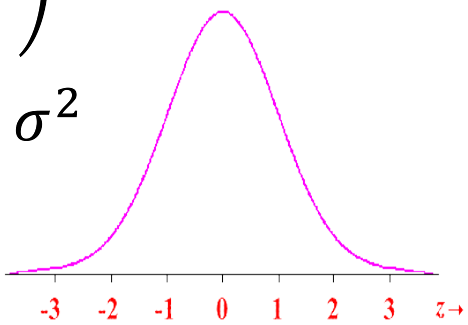
- Two random variables X and Y on the same experiment
  - " two throws of a dart into a unit interval "
- ***Joint distribution*** of X and Y
  assigns probability $\Pr(X \in I, Y \in J)$, for any two intervals I, J
- X and Y are independent if for all intervals I, J
  $$\Pr(X \le x, Y \le y) = \Pr(X \le x)\Pr(Y \le y)$$
  - equivalently: if events $\{X \le x\}$ and $\{Y \le y\}$ are independent
- Random variables $X, Y, Z, \ldots$ ***mutually independent*** if
  $$\Pr(X \le x, Y \le y, Z \le z, \ldots) = \Pr(X \le x)\Pr(Y \le y)\Pr(Z \le z)\ \ldots$$

Normal (Gaussian) Distribution

- Random variable $X \sim N(\mu, \sigma^2)$ defined by pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

  - two parameters: expectation μ and variance σ^2
  - "standard normal distribution": $N(0,1)$



- Nice properties:
  - If $X_1 \sim N(\mu_1, \sigma^2_1)$ and $X_2 \sim N(\mu_2, \sigma^2_2)$ are independent, then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma^2_1 + \sigma^2_2)$
  - Central Limit Theorem (informally):
    If $Y_1, \ldots, Y_n$ are IID RVs with finite variance,
    their average converges to a normal distribution as $n \to \infty$

Outline

- Basics: “discrete” probability
- Basics: “continuous” probability
- Concentration inequalities

Concentration inequalities

- **Setup:** $X_1, \dots, X_n$ random variables.
  (not necessarily identically distributed)
  $\bar{X} = \frac{X_1 + \cdots + X_n}{n}$ is the average, and $\mu = \mathbb{E}[\bar{X}]$
- Strong Law of Large Numbers:
  $$\Pr\left(\bar{X} \xrightarrow{n} \mu\right) = 1$$
- Want: $\bar{X}$ is ***concentrated*** around μ when n is large,
  i.e. that $|\bar{X} - \mu|$ is small with high probability.
  - $\Pr(|\bar{X} - \mu| \leq \text{"small"}) \geq 1 - \text{"small"}$
  - such statements are called "concentration inequalities"

Hoeffding Inequality (HI)

- High-prob. event: $\mathcal{E}_{\alpha,T} = \left\{ |\bar{X} - \mu| \leq \sqrt{\frac{\alpha \log T}{n}} \right\}$, $\alpha \geq 0$
- **<u>HI:</u>** Assume $X_i \in [0,1]$ for all i. Then

$$\Pr(\mathcal{E}_{\alpha,T}) \geq 1 - 2T^{-2\alpha}.$$

  - $\alpha = 2$ suffices for most applications in this course.
    T controls probability; can be the time horizon in MAB
  - this is a convenient re-formulation of HI for our purposes
    more "flexible" and "generic" formulation exists
- "Chernoff Bounds": special case when $X_i \in \{0,1\}$
- Relevant notation: $r = \sqrt{\frac{\alpha \log T}{n}}$ "confidence radius

  $[\mu - r,\ \mu + r\,]$ "confidence interval"

Hoeffding Inequality (extensions)

- **Recall**: $\mathcal{E}_{\alpha,T} = \left\{ |\bar{X} - \mu| \leq \sqrt{\frac{\alpha \log T}{n}} \right\}, \alpha \geq 0$
- **"HI for intervals":** Assume $X_i \in [a_i, b_i]$ for all i. Then

  $$\Pr(\mathcal{E}_{\alpha\beta, T}) \geq 1 - 2T^{-2\alpha}, \text{ where } \beta = \frac{1}{n}\sum_{i=1}^{n}(b_i - a_i)^2.$$
- **"HI for small variance":**
  Assume $X_i \in [0,1]$ and $Var(X_i) \leq v$ for all i. Then

  $$\Pr\left(\mathcal{E}_{\alpha v, T}\right) \geq 1 - 2T^{-\alpha/4}.$$

  as long as n is large enough: $\frac{n}{\log n} \geq \frac{\alpha}{9v}$.
- **"HI for Gaussians":**
  Assume X_i is Gaussian with variance $\leq v$.Then

  $$\Pr\left(\mathcal{E}_{\alpha v, T}\right) \geq 1 - 2T^{-\alpha/2}.$$

Concentration for non-independent RVs

- **Setup:** $X_1, \dots, X_n$ independent random variables in [0,1] (***not necessarily independent*** or identically distributed)
  $\bar{X} = \frac{X_1 + \cdots + X_n}{n}$ is the average
- **Assume:** there is a number $\mu_i \in [0,1]$ such that
  $$E(X_i | X_1 \in J_1, \dots, X_{i-1} \in J_{i-1}) = \mu_i$$
  for any intervals $J_1, \dots, J_{i-1} \subset \Re$.

for each $i = 1, \dots, n$

$\mu = (\mu_1 + \cdots + \mu_n)/n$

- **Let** $\mathcal{E}_\alpha = \left\{ |\bar{X} - \mu| \le \sqrt{\frac{\alpha \log T}{n}} \right\}, \alpha \ge 0$
- Then (corollary from "Azuma-Hoeffding inequality")
  $$\Pr\left(\mathcal{E}_{\alpha,T}\right) \ge 1 - 2T^{-\alpha/2}$$

Resources

- A survey on concentration inequalities
  by Fan Chung and Linyuan Lu (2010)
- Another survey on concentration inequalities
  by Colin McDiarmid (1998).
- Wikipedia
  - Hoeffding inequality
  - Azuma-Hoeffding inequality