How Do Computers Solve Geometric Problems?

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Outline

- Introduction
  - Algorithms
  - Computational Geometry
- Art Museum Problem
- The Cheapest Place to Find a Cheeseburger

Algorithms

What is an algorithm?

- A sequence of precise instructions
- Followed to reliably solve a problem
  - Proof needed
- Represent computer programs

Computational Geometry

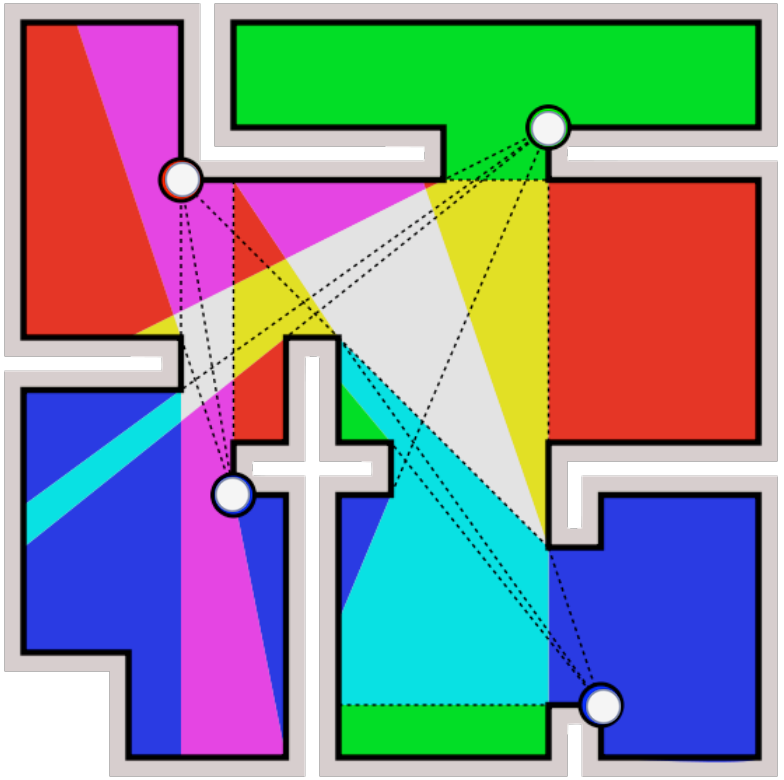
Study of geometry using computers and for computers

- Examples
  - Line segment intersections
  - Moving point collisions
  - Point clustering
- Applications
  - GPS systems
  - Navigation
  - Animation
  - Cell phone service

Art Museum Problem

Protect the art with the minimum number of cameras

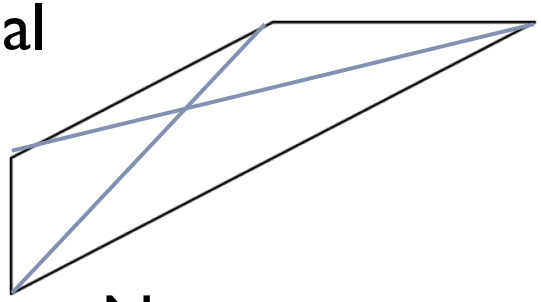
- Gallery represented by a simple polygon
- Polygon has n sides
- Cameras and art can be placed anywhere
- How many cameras are needed to view the whole area? (as a function of n)



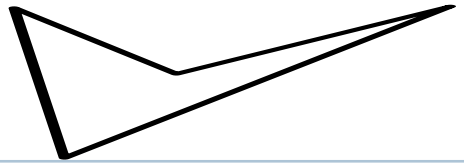
Art Museum Problem

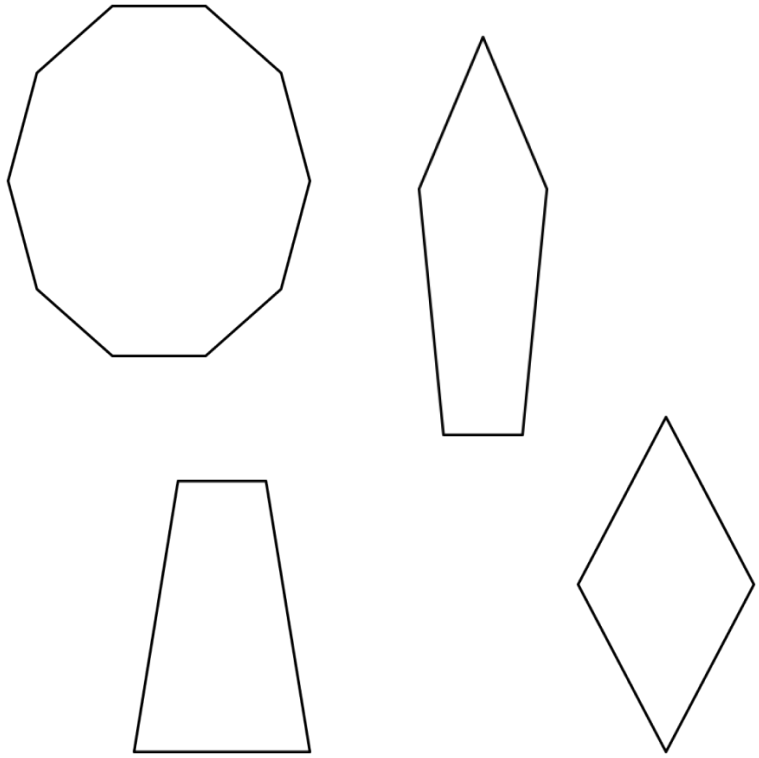
How many cameras are necessary?

- Only one camera needed for any convex polygon
- Convex: All diagonals are internal



- Concave: Not convex





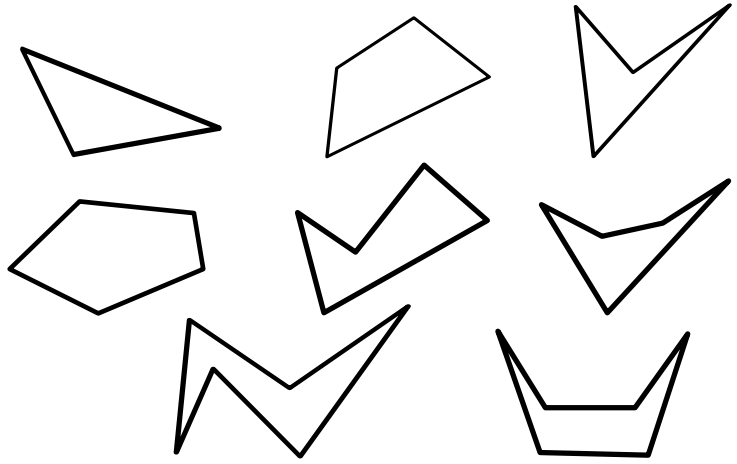
Art Museum Problem

How many cameras are necessary?

- For any polygon with n sides

sides	cameras
3	1
4	1
5	1
6	2

- Look at polygons with 3-6 sides
- Remember concave polygons



Art Museum Problem

What is the function c(n) for the number of cameras?

- For any polygon with n sides

sides	cameras
3	1
4	1
5	1
6	2

- $\lfloor x \rfloor$ rounds x down to the nearest integer

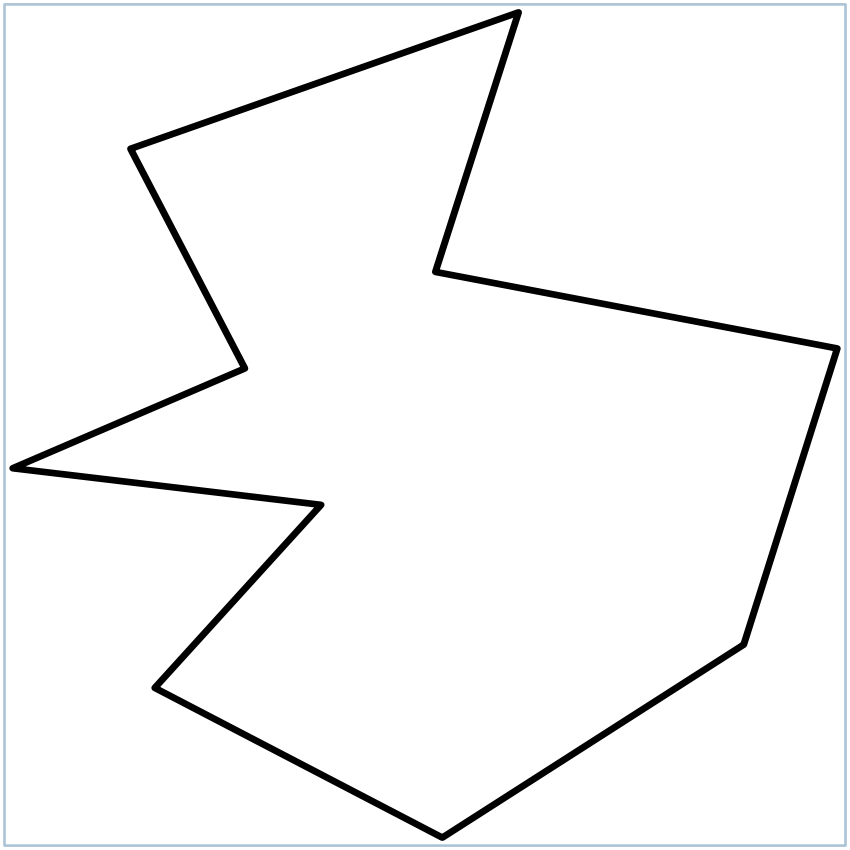
$$c(n) = \left\lfloor \frac{n}{3} \right\rfloor$$

- This shows c(n) cameras are necessary
- Will show c(n) cameras are sufficient

Art Museum Problem

How can we determine where to place the cameras?

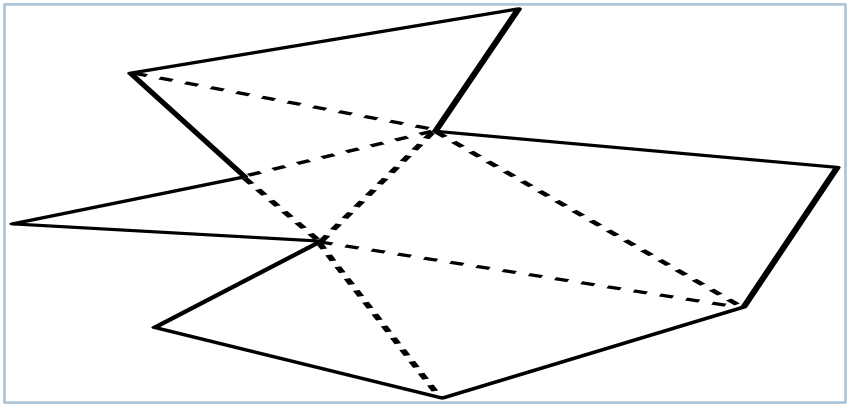
- 10-sided polygon
- 3 cameras allowed
  - Only one needed
- What do you think?
  - When is it easy to place one camera?

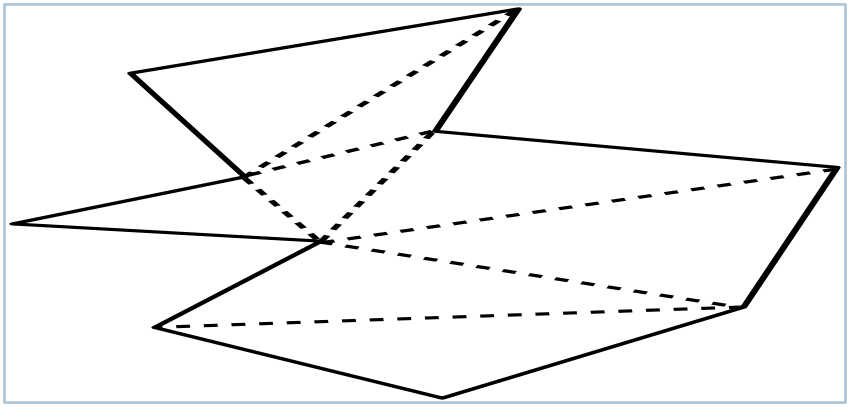


Art Museum Problem: Triangulation

What is a triangulation of a polygon?

- A partition of a polygon into triangles
- Add internal diagonals between vertices until no more can be added without crossing others
- There can be multiple ways to triangulate a polygon



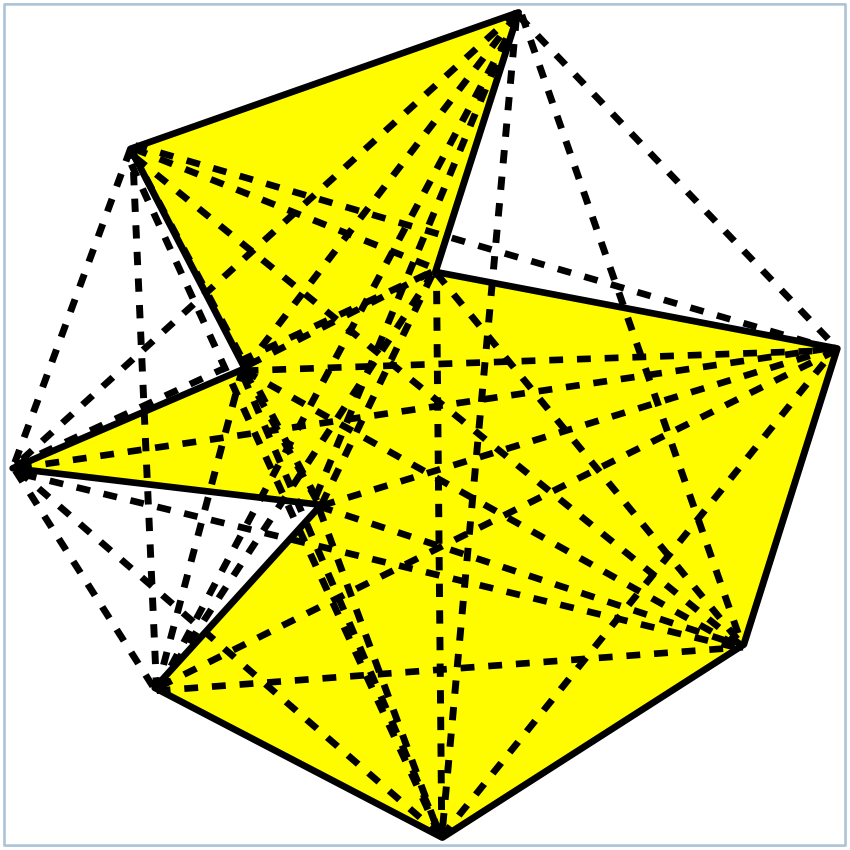


Art Museum Problem: Triangulation

How can we divide a polygon into triangular regions?

Algorithm:

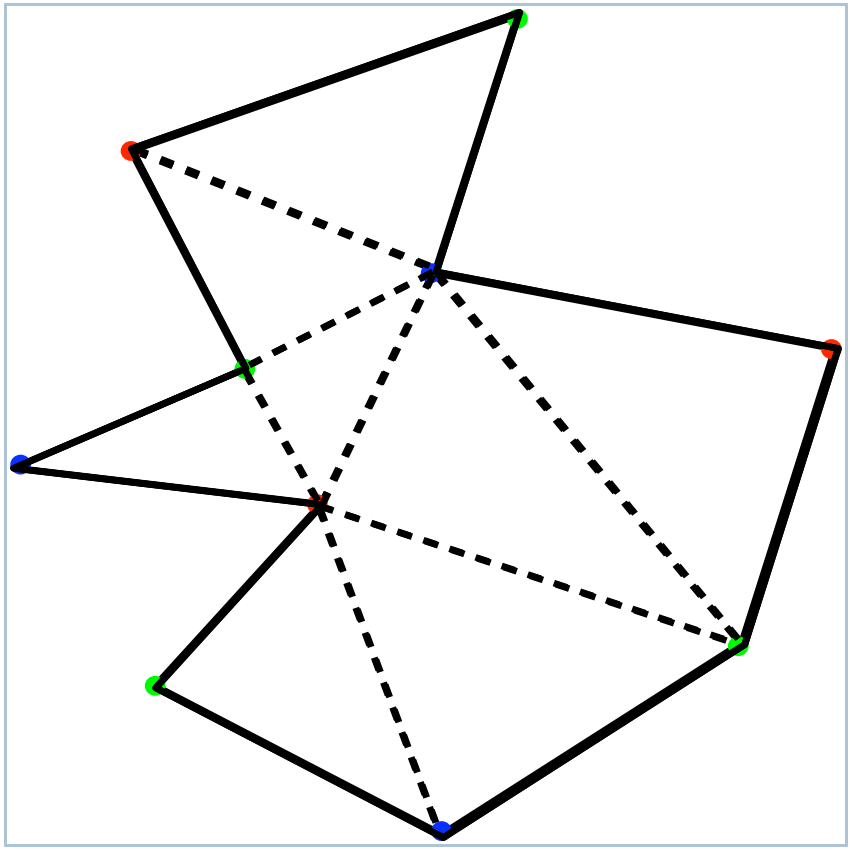
- Consider all diagonals
- Until no more diagonals can be inserted
  - Choose a diagonal that is
    1. Internal
    2. Does not cross any previously chosen diagonals
- Chosen diagonals give triangulation



Art Museum Problem

How can we automate camera placement?

- Algorithm
  - Triangulate the polygon
  - 3-color the vertices
    - No color can share an edge with itself
  - Choose the color that appeared least
    - No more than n/3 of that color
  - Put a camera on every vertex that color



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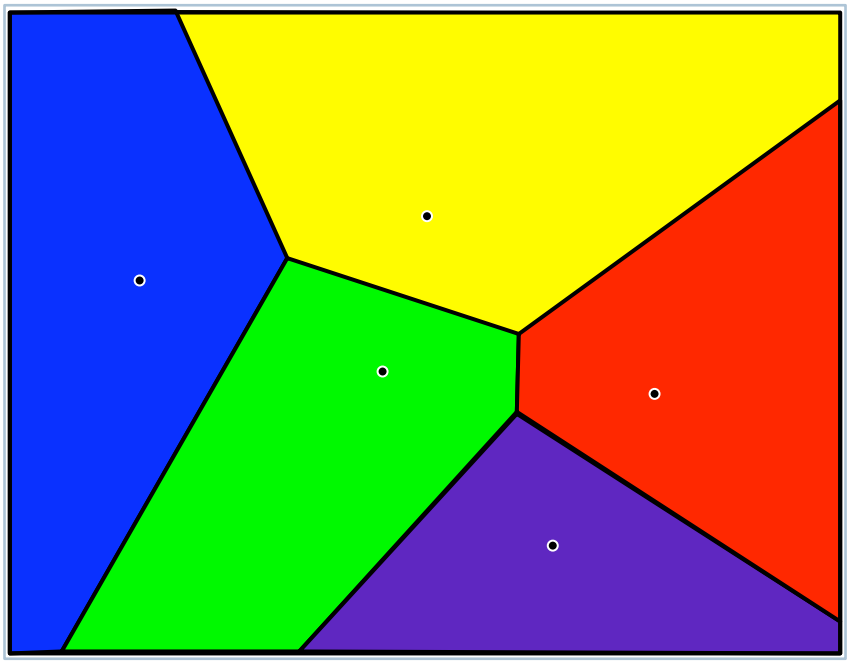
Voronoi Diagrams: Numb3rs clip

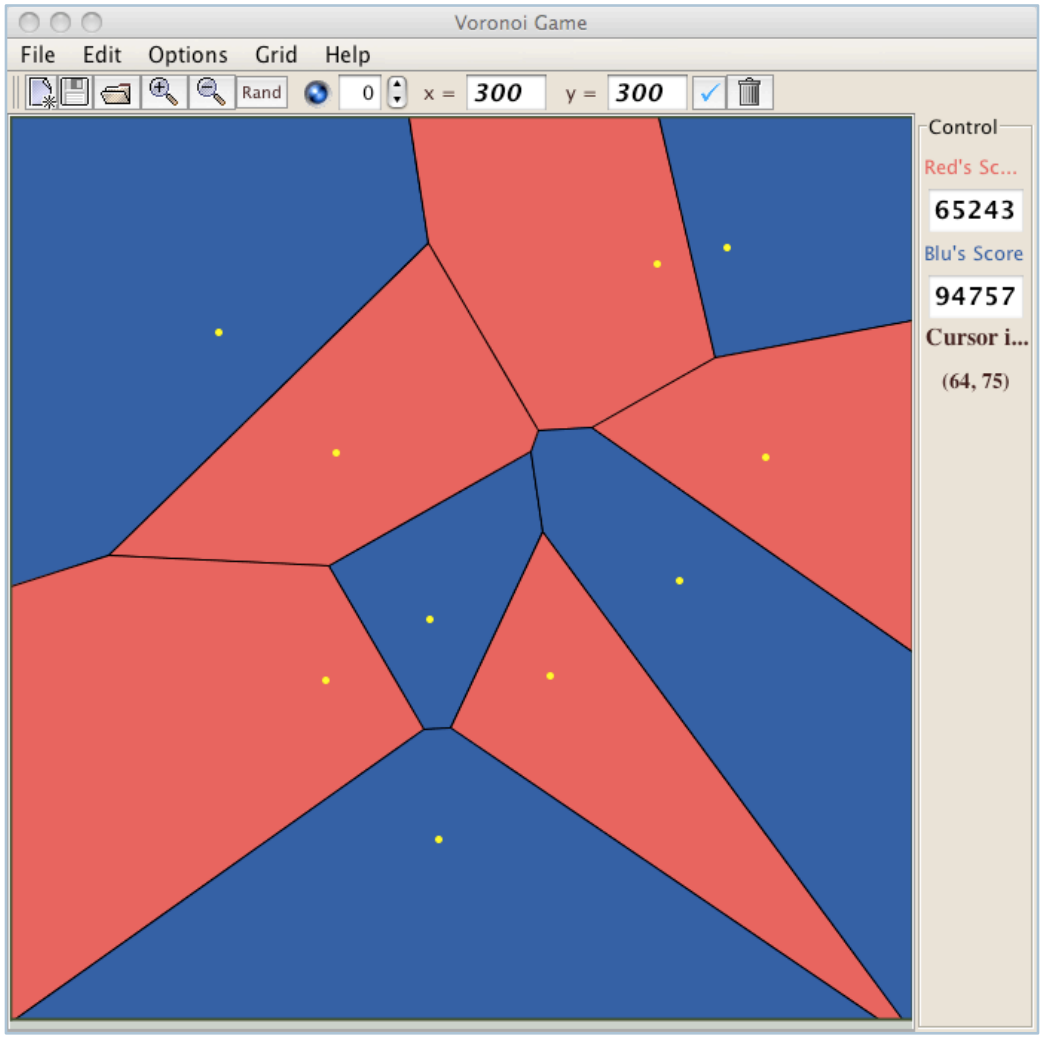
Voronoi Diagrams

Which site is closest to a given point?

- Divide space into regions in which all points are closer to the site representing the region than to any other site.
- Applications:
  - Cell phone towers
  - Fire station regions
  - Laziness



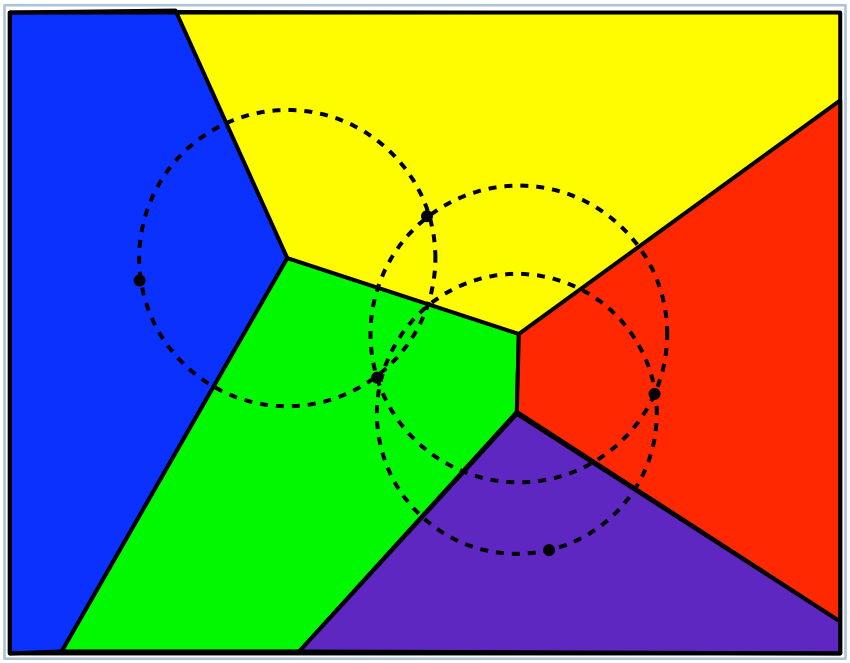
Voronoi Diagrams



Voronoi Diagrams

Circle property

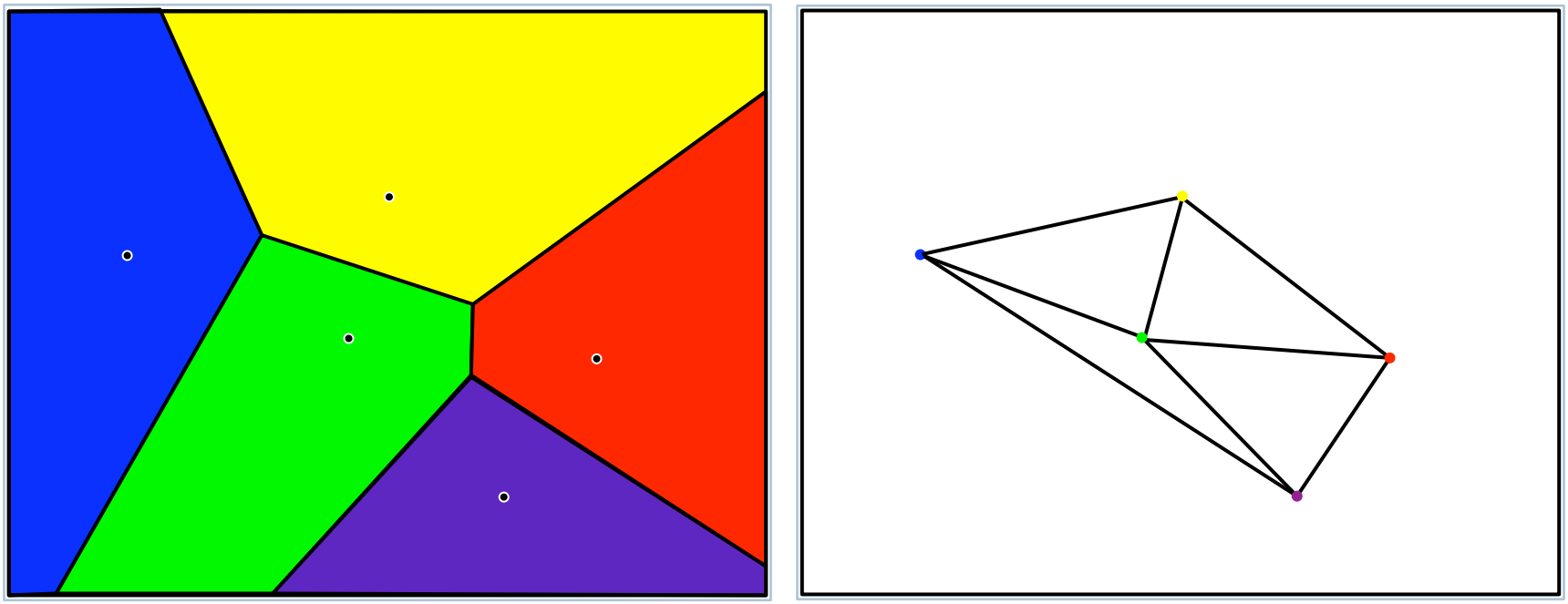
- Center a circle at the intersection of 3 region boundaries
- 3 sites on its perimeter
- No sites in its interior



Voronoi Diagrams: Delaunay Triangulations

Dual version of Voronoi diagrams:

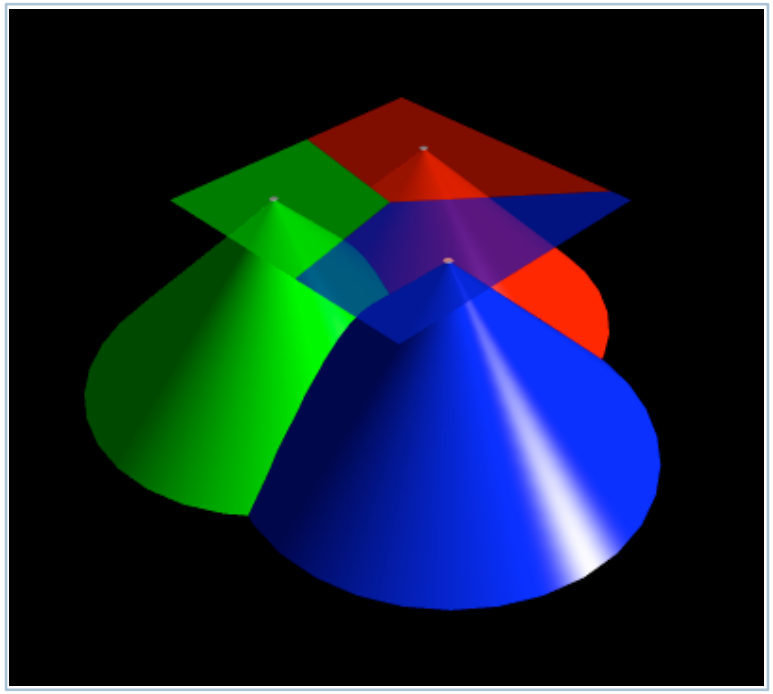
Create edges between adjoining cells' sites



Voronoi Diagrams

How can we determine the Vornoi diagram?

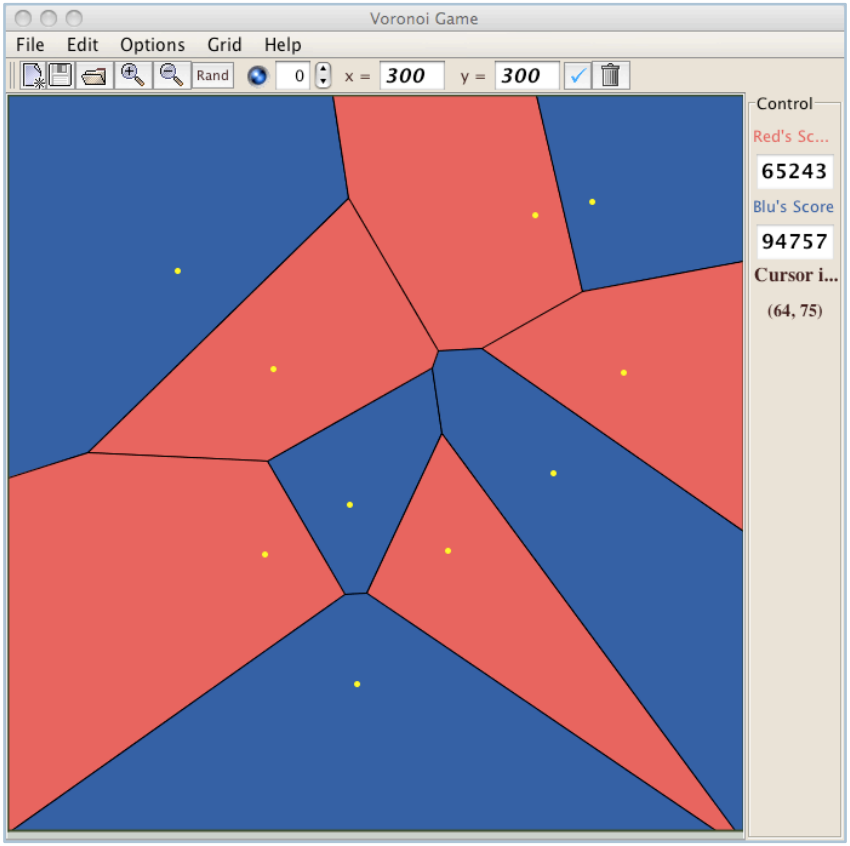
- Given n sites (points)
- Let each site grow cones at the same rate
- Cones are at 45° angles
- The flattened/projected view of the cones is the Voronoi diagram



Voronoi Game

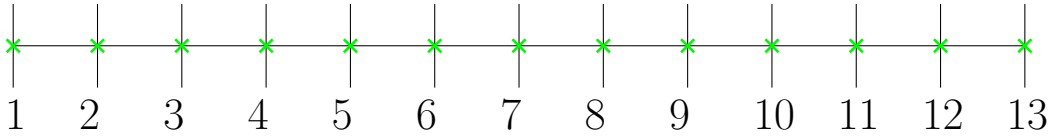
Try to serve the largest area

- Players alternate turns
- Each player gets a set number of sites to place
- Each turn for each player
  - The player places a site
  - The area is divided into Voronoi regions
- Player with the greatest area wins



Voronoi Game

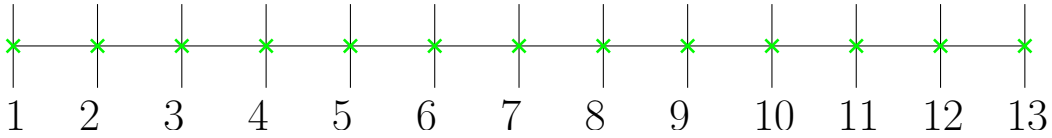
One-dimensional version



- Goal: Control the most items (green crosses)
- Rules:
  - Players alternate turns
  - Each player gets a set number of sites to place
  - On each turn, each player places a site on an integral location
  - Player with some site closest to the most items wins
    - Equally close sites are awarded to no-one

Voronoi Game

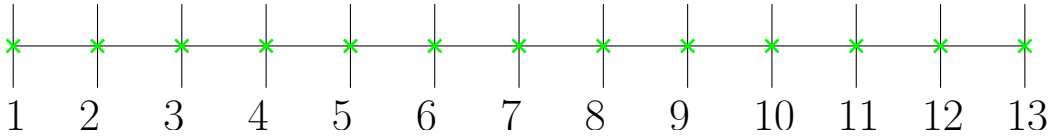
What's the best strategy?



- Play the game, changing the number of sites that you each place and who goes first or second.
- If you have to go second, how many sites do you want to be able to place?
  - 1 or 7 guarantee that you lose. Why?
  - Can you guarantee a tie or a win for a 2 site game if you get to go first? What if you have to go second?

Voronoi Game

2 round, one-dimensional version



- Control of both 4 and 10 wins the game
- The second player playing symmetrically guarantees a tie
- The first player can not guarantee a win

References (http://www.cs.umd.edu/~sorelle)

- General Computational Geometry
  - Mark de Berg, Otfried Cheong, Marc van Kreveld, and Mark Overmars. *Computational Geometry*. Springer-Verlag. 3rd edition, 2008.
- Art Museum Problem
  - Joseph O'Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, 1987

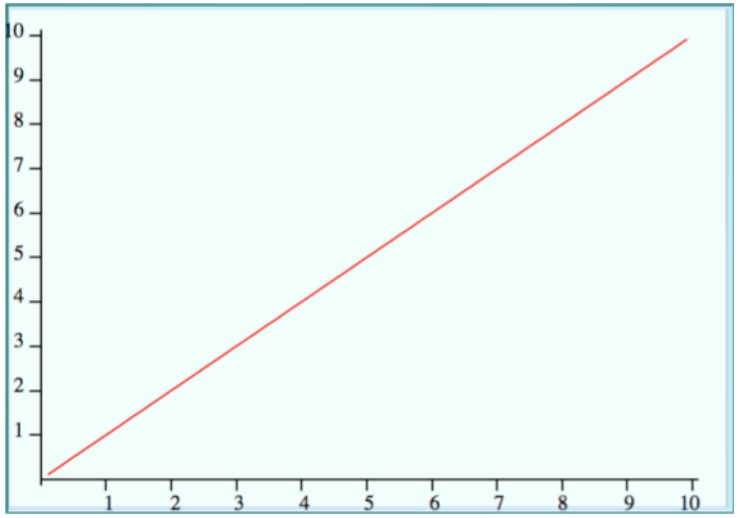
  http://maven.smith.edu/~orourke/books/ArtGalleryTheorems/art.html
- Voronoi Diagram and Game
  - http://www.voronoigame.com (description and applet)
  - http://home.dti.net/crispy/Voronoi.html (computer applet)
  - Dennis Shasha. *Dr. Ecco: Mathematical Detective*. Chapter 15: The Territory Game. Courier Dover Publications, 2004

Asymptotic Notation

If the input size is *n*, how long does the algorithm take?

- O(n)

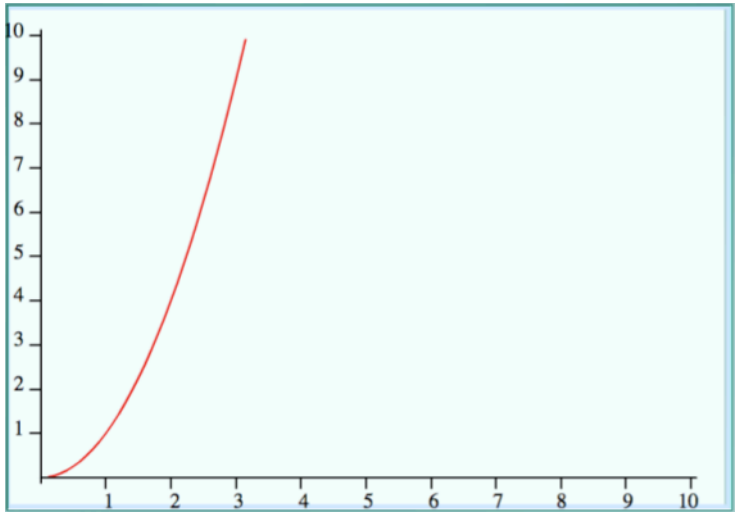
time



n

- $O(n^2)$

time



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