# Networks Chapter 5
# Structural Balance

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Social Networks Seminar

# Motivation

- The structure of a network isn't the full picuture
- Want information on the edges
  - Positive vs. negative
- Understand tension between people
  - Friends vs. enemies
- Structural balance illustrates local effects impacting global properties

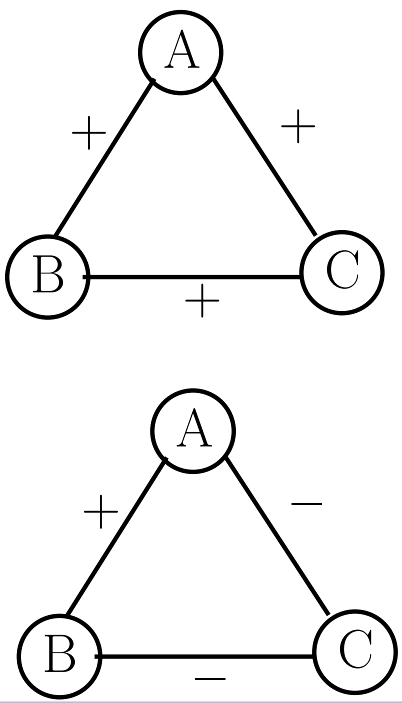
# Definitions

- Given a complete graph where nodes are people
- Label edges:
  - + friends
  - - enemies
- Model makes sense for small groups or famous nodes (like countries)

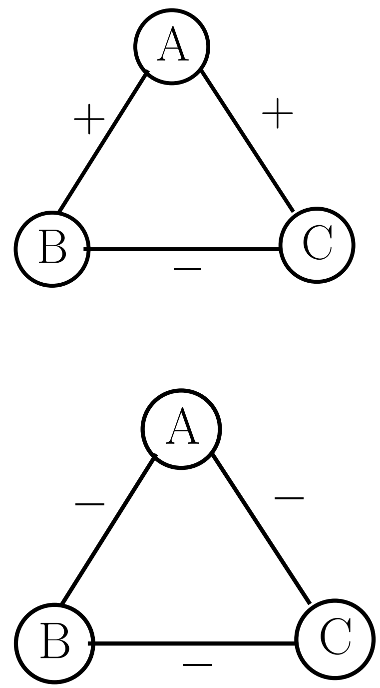
# Definitions

## Some 3 node configurations are psychologically more or less likely

- **Balanced**

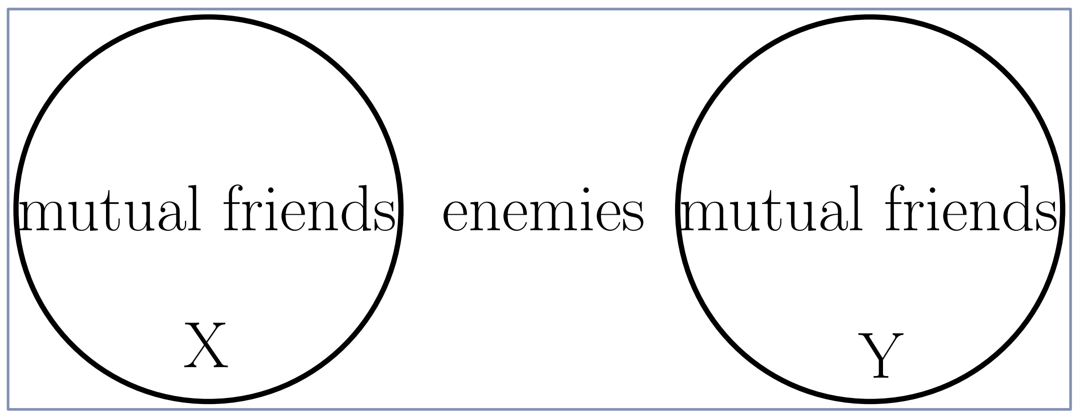



- **Unbalanced**



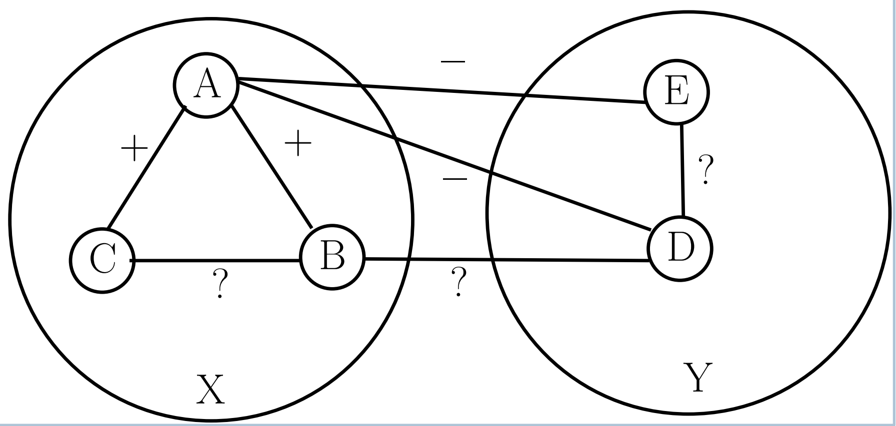
# Definitions

- A graph is balanced if all subsets of 3 nodes are balanced
- Examples:

# Cartwright-Harary Theorem (1950s)

- All balanced graphs look like one of those two examples
- Proof:
  - Suppose we have some balanced graph.
  - If no negative edges, we're done. Assume some negative edge.
  - Pick some node A. Let X be the friends of A. Let Y be the enemies of A.
  - Every two nodes in X are friends
  - Every two nodes in Y are friends
  - Every pair of nodes in X and Y are enemies



# International relations

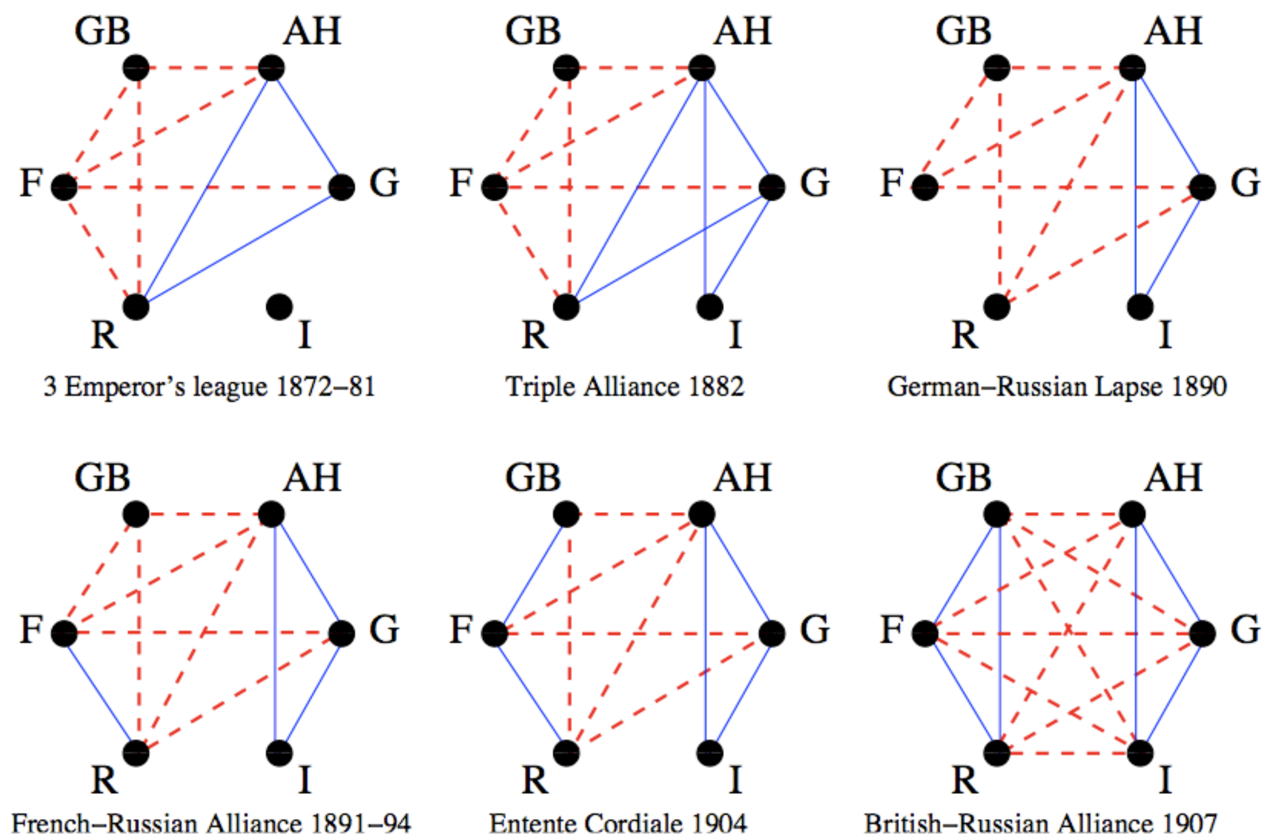



Fig. 10. Evolution of the major relationship changes between the protagonists of World War I from 1872–1907. Here GB = Great Britain, AH = Austria-Hungary, G = Germany, I = Italy, R = Russia, and F = France.

Antal, Krapivsky, and Redner. Social balance on networks: The dynamics of friendship and enmity. Physica D. 2006

# Extensions

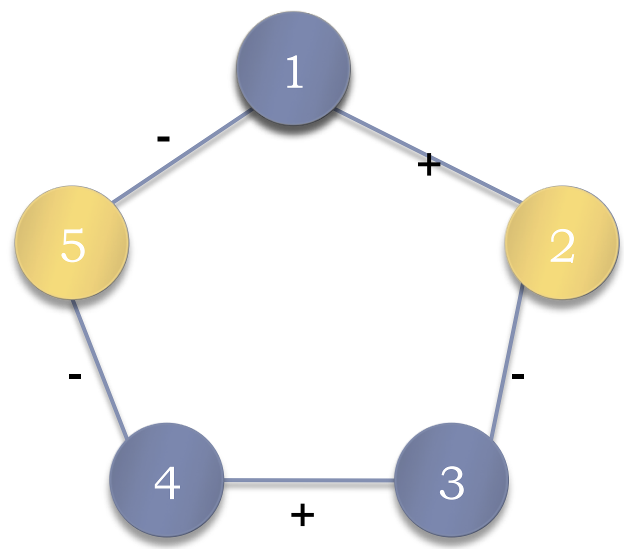
- Graph is balanced if almost all triangles are balanced
  - Globally: two groups with almost mutual friendship internally, and almost mutual antagonism between
- Some edges might be indifferent
  - Global conditions exist here too
- Evolution over time
  - Start with random labeling
  - Find unbalanced triangle and flip an edge so that it's balanced
  - Resembles models for physical systems minimizing energy

# Generalized Definition: Non-Complete Graphs

- Edges may only be + or -, but not all edges exist.
- Definition 1 (Local): Possible to fill in missing edges so that complete graph is balanced
- Definition 2 (Global): Possible to divide nodes into sets X and Y as defined previously
- Definition 1 = Definition 2:
  - 1=>2: Fill in all the edges. Use Cartwright-Harary.
  - 2=>1: Partition. Fill in edges to maintain X and Y properties.
- But how can we check if a graph is balanced? …

# Characterization of General Signed Graphs (Cartwright and Harary)

- What prevents a graph from being balanced?




- Claim: A signed graph is balanced <u>if and only if</u> it contains no cycle with an odd number of negative edges.

# Proof of Claim

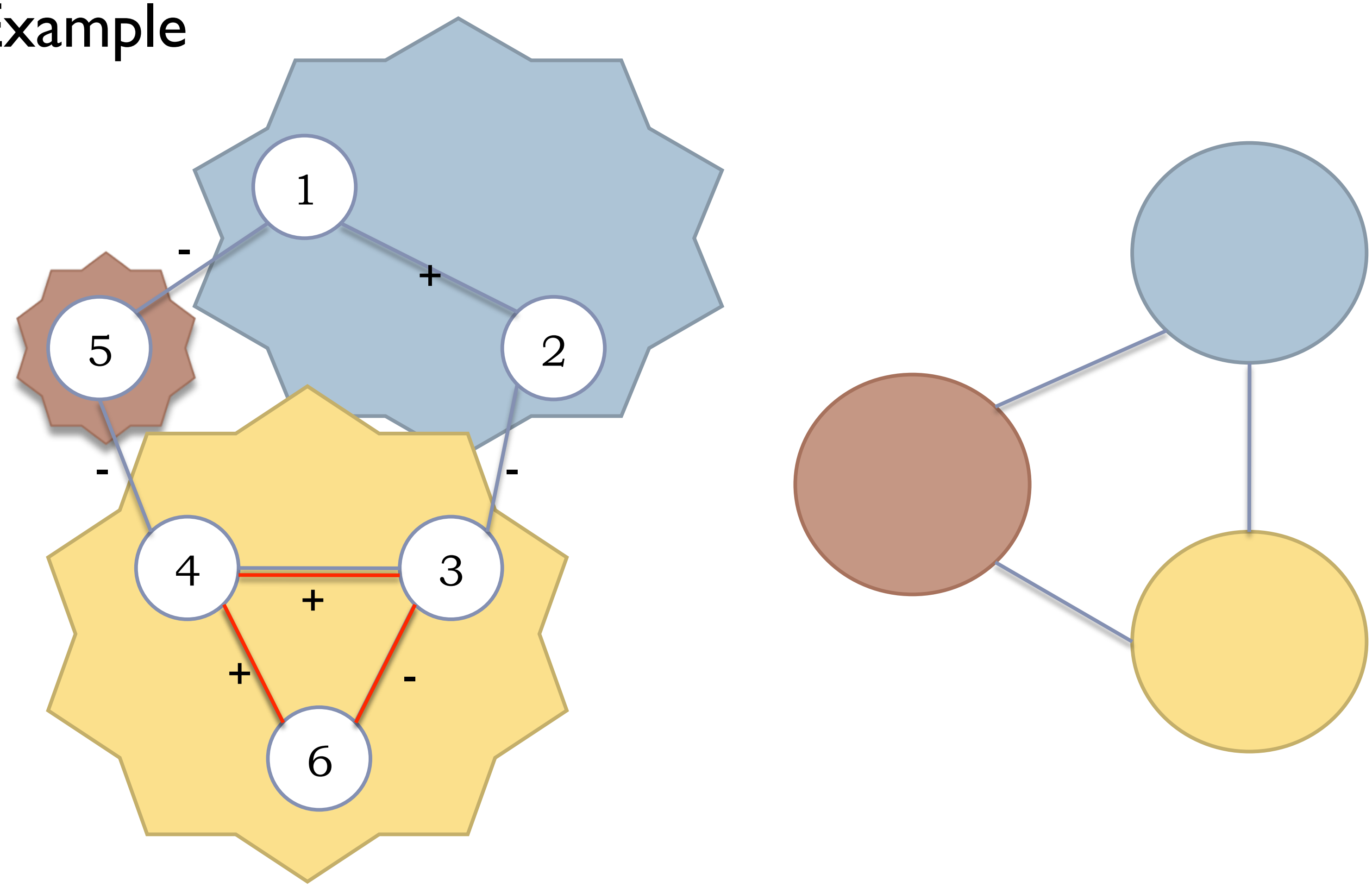
- Claim: A signed graph is balanced <u>if and only if</u> it contains no cycle with an odd number of negative edges
- Will give a method that either finds sets X and Y or finds a cycle with an odd number of negative edges
  - This will prove the claim
- First: Transform graph so that it only has negative edges
- Second: Solve the problem on the reduced graph

# Proof: Step 1 – Graph Transformation

- Any two nodes connected by a positive edge must be in the same partition
- Create supernodes that contain all nodes connected by positive edges
- If any supernode has internal negative edges, this creates a cycle with an odd (1) number of negative edges
  - Both edge vertices were added by only following positive edges
- Create edges between the supernodes if there is any edge connecting two member nodes in the original graph

# Proof: Step 1 – Graph Transformation

- Example

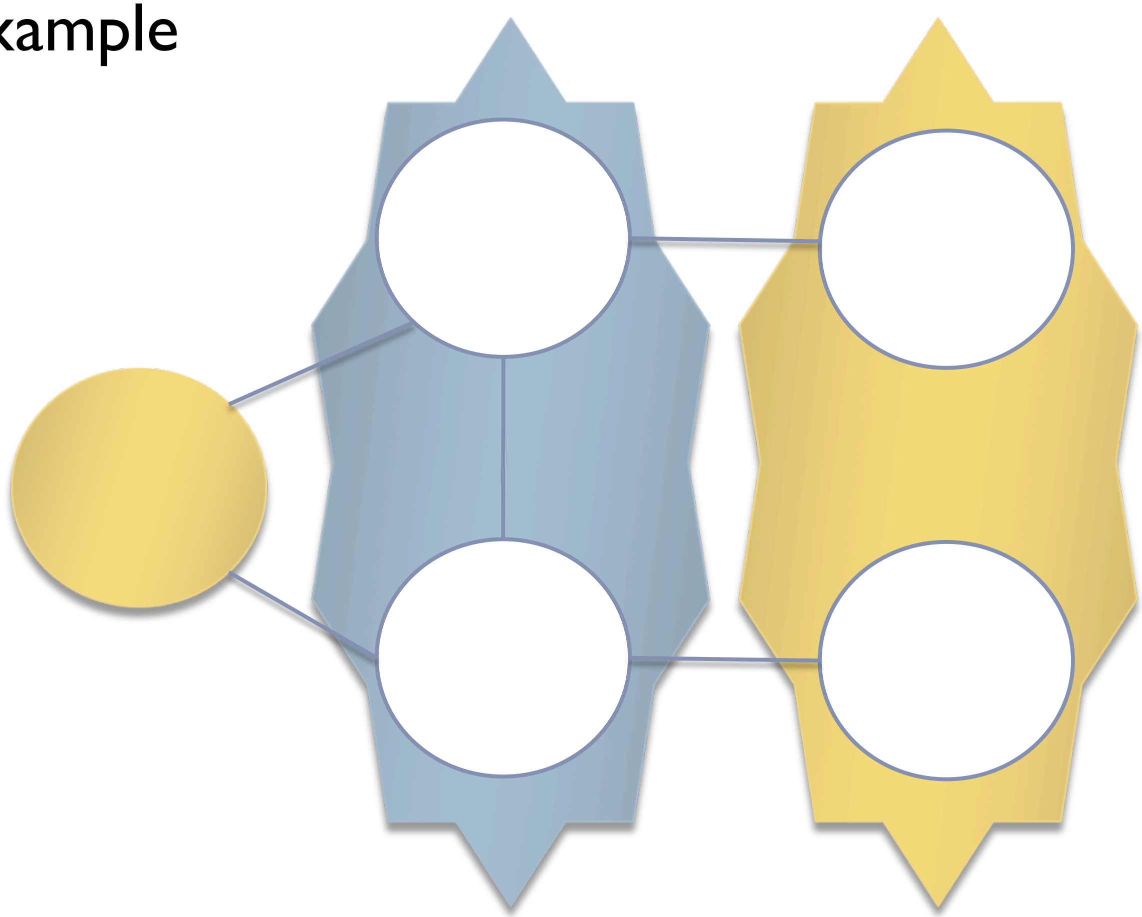


# Proof: Step 2 – Partition Reduced Graph

- Recall: we will either label the partitions or find a cycle with an odd number of negative edges
  - First case: Label member nodes the same as their supernodes
  - Second case: Make cycle by using positive edges in supernodes
- Partition the graph (determine if it's bipartite)
  - Pick some node A
  - Use breadth-first search, putting nodes that are an odd distance away from A in one partition and even distance nodes in another.
  - If there is an edge between nodes in a partition, there is a cycle, otherwise we have a partition.

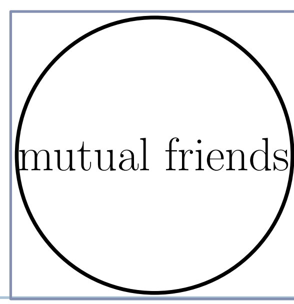
# Proof: Step 2 – Partition Reduced Graph

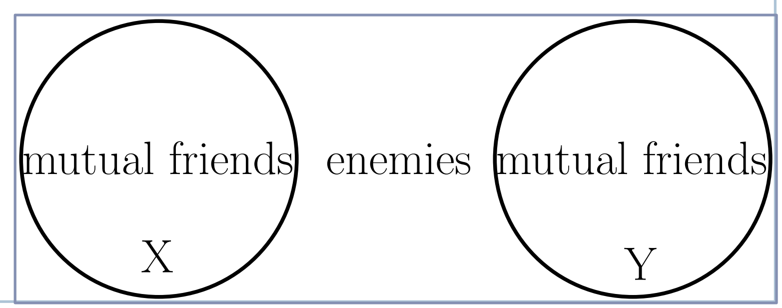
- Example



# Generalized Definition: Fraction-based Balance

- Definition: At least $1-\varepsilon$ of all triangles are balanced
  - $0 \leq \varepsilon < 1/8$
- Let $\delta = \sqrt[3]{\varepsilon}$
- Claim: If a graph is balanced than one of these is true:
  - There is a subset of $1-\delta$ nodes in which at least $1-\delta$ of all pairs are friends
  - The graph can be divided into X and Y as before so that $1-\delta$ of all edges in X, $1-\delta$ in Y, and $1-\delta$ between have the correct property



# Proof of Claim: Overview and Preliminaries

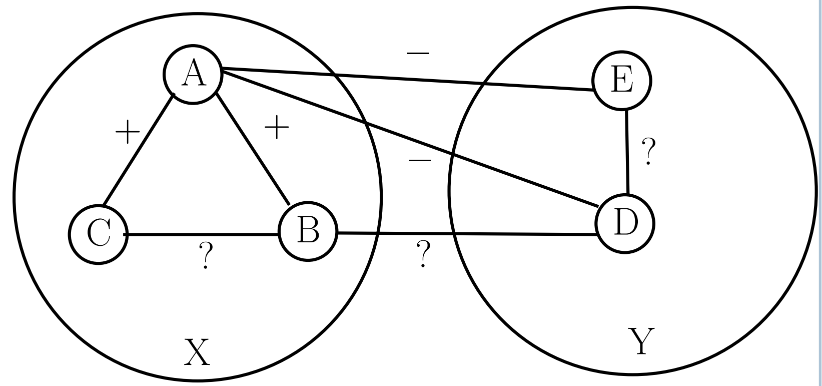
- Assume graph is balanced, show it has one of the two forms
- As before, define two sets X and Y to be friends and enemies of some node A
  - But what if A happens to be involved in too many unbalanced triangles?
  - A must be a "good" node
- Given a complete graph with N nodes
  - How many edges? N choose 2: N(N-1)/2
  - How many triangles? N choose 3: N(N-1)(N-2)/6

# Proof of Claim: Find a "Good" Node

- There are at most $\varepsilon N(N-1)(N-2)/6$ unbalanced triangles
- Find each node's *weight*: the number of unbalanced triangles it participates in
  - Total weight of all nodes: $\varepsilon N(N-1)(N-2)/2$
  - Average weight of all nodes: $\varepsilon (N-1)(N-2)/2$
- Pick some node A with weight less than the average
  - They actually use $\varepsilon N^2/2$

# Proof of Claim: Characterize the Graph

- Let X be all the friends of A and Y the enemies of A
- Since A is involved in less than $\varepsilon N^2/2$ unbalanced triangles, properties within and between X and Y fail < $\varepsilon N^2/2$ times (out of what total?)

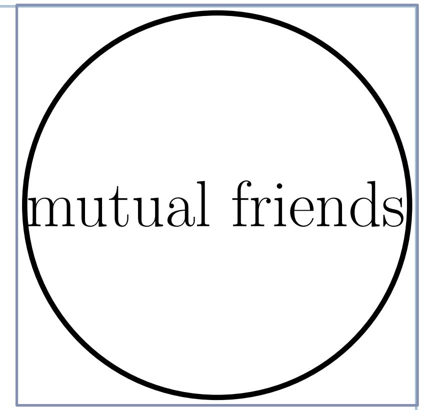



- Determine if one of X or Y is "small"
  - If so, most nodes are in a single mutually friendly group
  - If not, the nodes are divided into two mutually friendly groups who are mutual enemies

# Proof of Claim: Characterize the Graph

- Let x = |X| and y = |Y|
- Suppose first that X is big, $x \geq (1-\delta)N$
  - By choices of $\varepsilon$ and $\delta$, $x > \frac{1}{2}N$
  - X has $x(x-1)/2$ internal edges
  - X has at least $N^2/8$ internal edges
  - Fraction of negative internal edges is $\dfrac{\varepsilon N^2/2}{N^2/8} < \delta$
    (A implies numerator, $\varepsilon = \delta^3$ and $\delta < \frac{1}{2}$)

  - X contains at least $(1-\delta)N$ of the nodes
  - At least $(1-\delta)N$ of the pairs in X are friends
- Same arguments hold if Y is big

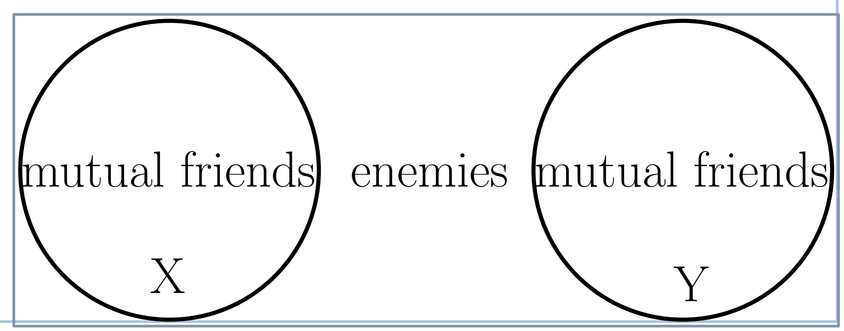


# Proof of Claim: Characterize the Graph

- Suppose neither X nor Y is big: $x<(1-\delta)N$ and $y<(1-\delta)N$
  - There are xy edges between, $xy \le \delta N^2/2$
  - At most $\varepsilon N^2/2$ of these are positive.
  - Fraction positive: $\dfrac{\varepsilon N^2/2}{\delta N^2/2} < \delta$
  - There are $x(x-1)/2$ internal edges in X (same for Y)
  - Since $x>\delta N$, this total is at most $\delta^2 N^2/2$
  - Since A is good, at most $\varepsilon N^2/2$ negative internal edges

$$\frac{\varepsilon N^2/2}{\delta^2 N^2/2} = \delta$$

- This proves the claim.



Thanks!
Questions?