Property B: Two-Coloring Non-Uniform Hypergraphs

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Abstract

The following is a classical question of Erdős (*Nordisk Matematisk Tidskrift,* 1963) and of Erdős and Lovász (*Colloquia Mathematica Societatis János Bolyai,* vol. 10, 1975). Given a hypergraph $\mathcal{F}$ with minimum edge-size k, what is the largest function $g(k)$ such that if the expected number of monochromatic edges in $\mathcal{F}$ is at most $g(k)$ when the vertices of $\mathcal{F}$ are colored *red* and *blue* randomly and independently, then we are guaranteed that $\mathcal{F}$ is two-colorable? Duraj, Gutowski and Kozik (*ICALP 2018*) have shown that $g(k) \geq \Omega(\log k)$. On the other hand, if $\mathcal{F}$ is k-uniform, the lower bound on $g(k)$ is much higher: $g(k) \geq \Omega(\sqrt{k/\log k})$ (Radhakrishnan and Srinivasan, *Rand. Struct. Alg.*, 2000). In order to bridge this gap, we define a family of locally-almost-uniform hypergraphs, for which we show, via the randomized algorithm of Cherkashin and Kozik (*Rand. Struct. Alg.*, 2015), that $g(k)$ can be much higher than $\Omega(\log k)$, e.g., $2^{\Omega(\sqrt{\log k})}$ under suitable conditions.

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1 Introduction

A classical question of Erdős and of Erdős and Lovász is as follows [7, 9]. Given a hypergraph $\mathcal{F}$, let us define its minimum edge-size k as the (asymptotic) parameter of interest. Note that if the vertices of $\mathcal{F}$ are colored *red* and *blue* randomly and independently, then the expected number of of monochromatic edges in $\mathcal{F}$ is

$$M(\mathcal{F}) \doteq \sum_{f \in \mathcal{F}} 2^{1-|f|}.$$

(We view $\mathcal{F}$ as a collection of hyper-edges, hence the notation "$f \in \mathcal{F}$". Also, the constant multiplier "2" in the "$2^{1-|f|}$" is often not very important in our context, and we will typically study the sum $\sum_{f \in \mathcal{F}} 2^{-|f|}$.) Then, what is the largest function $g(k)$ such that if $M(\mathcal{F}) \leq g(k)$, then we are guaranteed that $\mathcal{F}$ is two-colorable? $\mathcal{F}$ was defined to have *Property B* by Erdős when it is two-colorable – in honor of F. Bernstein, who had considered the problem earlier [3] – and such questions on sufficient conditions for $\mathcal{F}$ to possess Property B have been studied quite a bit since the 1970s. In this work, we show improved sufficient conditions when $\mathcal{F}$ is *locally-almost-uniform* – specifically, λ-*approximately-uniform* as defined later – for a large range of the local-uniformity λ.

We start by reviewing prior work. A simple union bound shows that any $g(k) < 1$ will suffice, but the question is whether $g(k)$ can tend to infinity as k grows, and how fast g can grow. Beck [2] was the first to show that g can be allowed to grow with k, by proving that $g(k) \geq \Omega(\log^* k)$. The next improvement came recently, when Duraj, Gutowski and Kozik showed that $g(k) \geq \Omega(\log k)$ [6]. (If $\mathcal{F}$ is simple, i.e., if any two distinct edges intersect in at most one vertex, then $g(k) \geq \Omega(\sqrt{k})$ [11].) Much better lower bounds on $g(k)$ are known if $\mathcal{F}$ is k-uniform: $g(k) \geq \Omega(\sqrt{k/\log k})$ here [10] – see Cherkashin and Kozik [5] for a simpler proof of this result. On the other hand, $g(k) \leq O(k^2)$, even for k-uniform hypergraphs (Erdős [8]). It has been conjectured for long that $g(k)$ could be $\Theta(k)$, at least for uniform hypergraphs [9]. This remains a tantalizing open problem. Hypergraph two-coloring has also been studied for models of random hypergraphs, and from the viewpoint of inapproximability (see, e.g., [1, 4]).

Note the large gap between the lower bound of $\Omega(\log k)$ [6] and the lower bound of $\Omega(\sqrt{k/\log k})$ that holds for k-uniform hypergraphs [10, 5]. How can we bridge this gap? One approach is to study "how much uniformity" we need in order to get good lower bounds on $g(k)$: to this end, we define a family of locally-almost-uniform hypergraphs, for which we show that $g(k)$ can be much higher than $\Omega(\log k)$. Our randomized algorithm is the same as that of Cherkashin and Kozik [5], but our analysis is different. For $\lambda \geq 1$, we say that $\mathcal{F}$ is λ-approximately-uniform if

$$\max_{f,f' \in \mathcal{F}:\ |f \cap f'|=1} \frac{|f'|}{|f|} \leq \lambda.$$

That is, we require the local-almost-uniformity property that any two edges that intersect in exactly one vertex, have their size-ratio bounded by λ. (Note that this is asking less than requiring that any two intersecting edges have their size-ratio bounded by λ.)

Let $\exp(x)$ denote e^x. The following definition and lemma will be useful in the context of Theorem 3.

▶ **Definition 1.** ***(Parameter*** γ***)*** *For $\lambda, \alpha, k \geq 1$ such that $\alpha, \lambda \leq k$, we define $\gamma(k, \alpha, \lambda) \geq 0$ by its square:*

$$\gamma(k,\lambda,\alpha)^2 \doteq \min_{\lambda' \in [1,\lambda]} 2((\lambda'-1)k+1)\left[\left(1+\frac{\alpha}{k}\right)^{(\lambda'-1)k+1} - \left(1-\frac{\alpha}{\lambda' k}\right)^{(\lambda'-1)k+1)}\right]^{-1}.$$

Note that $\gamma(k, 1, \alpha) = \sqrt{k/\alpha}$; for larger values of λ, we will use the following lower bound on $\gamma(k, \alpha, \lambda)^2$ when we apply our main theorem to specific cases.

▶ **Lemma 2.** $\gamma(k,\lambda,\alpha)^2 \geq \left(\frac{k}{\alpha}\right)\exp(-\alpha\lambda)$.

Proof. We provide a proof in the appendix. ◀

We next present our main theorem followed by some of its consequences, following which we develop its proof.

▶ **Theorem 3.** *Let k be a positive integer and let $\lambda, \alpha \geq 1$ such that $2\alpha^2\lambda \leq k$. Let $\gamma(k, \lambda, \alpha)$ be as in Definition 1. Let $\mathcal{F}$ be any λ-approximately-uniform hypergraph with $k = \min_{f \in \mathcal{F}} |f|$. Then, $\mathcal{F}$ is two-colorable if*

$$\sum_{f \in \mathcal{F}} 2^{-|f|} \leq \frac{1}{4}\min\{\exp(\alpha), \gamma(k,\lambda,\alpha)\};$$

furthermore, such a coloring can be obtained in randomized polynomial time via the algorithm of Cherkashin and Kozik [5].

Note that the above theorem yields the bound obtained by the authors [10] for uniform hypergraphs. Indeed, if we set $\lambda = 1$ and $\alpha = \frac{1}{2}\ln\frac{k}{\ln k}$ (recall that $\gamma(k, 1, \alpha) = \sqrt{k/\alpha}$), the above theorem implies that every k-uniform hypergraph with at most $(1/4)\sqrt{k/\ln k} \times 2^k$ edges is two-colorable – the constant $1/4$ can be improved slightly. This is not surprising, for the randomized algorithm we use to establish Theorem 3 reduces to the algorithm of Cherkashin and Kozik [5], and therefore yields the same bound with the same constant for k-uniform hypergraphs. (A minor subtlety is that $\lambda = 1$ does not imply k-uniformity: we can have $\lambda = 1$ and still allow two edges that intersect at two vertices or more, to have different sizes. However, such pairs of edges can often be ignored, as shown by the analyses of [10, 5] for uniform hypergraphs.)

For larger λ, we may use Lemma 2 to conclude that the hypergraph is two-colorable provided (for some choice of $\alpha \geq 1$)

$$\sum_{f\in\mathcal{F}} 2^{-|f|} \leq \frac{1}{4}\min\{\exp(\alpha), \sqrt{k/\alpha}\exp(-\alpha\lambda/2)\}.$$

Some illustrative examples (assume k is large):

- if $\lambda = 10$ and we set $\alpha = (\ln(k/\ln k))/12$, then we conclude that such a 10-approximately uniform hypergraph is two-colorable whenever

$$\sum_{f\in\mathcal{F}} 2^{-|f|} \leq \left(\frac{1}{4}\right)\left(\frac{k}{\ln k}\right)^{1/12};$$

- if $\lambda = \sqrt{\ln k}$ and we set $\alpha = \sqrt{\ln k}/2$, then we conclude that a λ-approximately-uniform hypergraph is two-colorable whenever

$$\sum_{f\in\mathcal{F}} 2^{-|f|} \leq \frac{1}{4}\exp(\sqrt{\ln k}/2);$$

- in general, we get nontrivial results for all $\lambda = o(\ln k)$.

2 Proof of the main result

Fix a hypergraph $\mathcal{F}$ satisfying the assumptions of Theorem 3. Let us use the following two-step randomized strategy due to Cherkashin and Kozik [5] that starts with all vertices uncolored.

Step 1: To each vertex v of the hypergraph independently assign a uniformly-random delay $\eta(v)$ from $[0, 1]$.

Step 2: One by one, color the vertices using colors $\{$*blue*, *red*$\}$ in increasing order of their delays. Color a vertex v, when its turn comes, *red* if there exists some edge e containing v such that v is the last vertex to be colored in e, and such that all other vertices in e have already been colored *blue*; else color v *blue*.

As in Cherkashin and Kozik [5], we have the following observation: if an edge is left monochromatic at the end, then all its vertices must be colored *red*. In particular, suppose the vertices in f were colored *red* in the order $v_1, v_2, \ldots, v_r$, then there must be an edge e such that

$$|e \cap f| = 1, \text{ and } v_1 \text{ is the last vertex to be colored in } e.$$

In such a case, we say that f *blames* e.

Specifically, f blames e iff the following three conditions hold:

- $|e \cap f| = 1$ with $e \cap f = \{v\}$, say;
- $\eta(v) > \eta(u)$ for all other vertices u in e;
- $\eta(v) < \eta(w)$ for all other vertices w in f.

Thus, the probability that the above algorithm fails to two-color the hypergraph is at most

$$\Pr[\exists e, f : e \text{ blames } f]. \tag{1}$$

It is tempting to note that

$$\Pr[e \text{ blames } f] = \frac{(|e|-1)! \cdot (|f|-1)!}{(|e|+|f|-1)!}$$

and apply the union bound

$$\Pr[\exists e, f : e \text{ blames } f] \le \sum_{(e,f)} \Pr[e \text{ blames } f].$$

However, a sum such as in the RHS of this union bound can be too large. Cherkashin and Kozik [5] suggest a nuanced approach in a similar situation; we adapt their approach to obtain a better bound for (1). The following claim is the main technical contribution of this work. Recall that $|e|/|f|,\ |f|/|e| \le \lambda$ whenever e blames f.

▷ Claim 4.

$$\Pr[\exists e, f : e \text{ blames } f] \le 2\sum_{e} 2^{-|e|} \exp(-\alpha) + 4 \sum_{(e,f):|e\cap f|=1} 2^{-|e|-|f|}\gamma(k,\lambda,\alpha)^{-2}.$$

Let us assume this claim and complete the proof of our theorem.

Proof of Theorem 3. By the assumption in our theorem, we have

$$\sum_{f\in\mathcal{F}} 2^{-|f|} \le \frac{1}{4}\min\{\exp(\alpha), \gamma(k,\lambda,\alpha)\}.$$

So,

$$2\sum_{e} 2^{-|e|} \exp(-\alpha) \le \frac{1}{2},$$

and

$$4 \sum_{(e,f):|e\cap f|=1} 2^{-|e|-|f|}\gamma(k,\lambda,\alpha)^{-2} \le 4\left(\sum_{e} 2^{-|e|}\gamma(k,\lambda,\alpha)^{-1}\right)^2 \le \frac{1}{4}.$$

It follows from Claim 4 that the algorithm of Cherkashin and Kozik, outlined above, properly two-colors the hypergraph with probability at least $\frac{1}{4}$. The theorem follows from this. ◀

We now establish Claim 4.

Proof of Claim 4: For each edge e, let

$$\delta(e) \doteq \alpha/(2|e|).$$

Fix e and f with $e \cap f = \{v\}$, and define events

$$\begin{aligned}\mathcal{E}_1(e) &\equiv \forall u \in e : \eta(u) < \frac{1}{2} - \delta(e);\\ \mathcal{E}_2(f) &\equiv \forall w \in f : \eta(w) > \frac{1}{2} + \delta(f);\\ \mathcal{E}_3(e,f) &\equiv (f \text{ blames e}) \text{ and } \neg\mathcal{E}_1(e) \text{ and } \neg\mathcal{E}_2(f).\end{aligned}$$

Then, we have

$$(f \text{ blames } e) \subseteq \mathcal{E}_1(e) \cup \mathcal{E}_2(f) \cup \mathcal{E}_3(e,f). \tag{2}$$

We bound the probability of each of these three events separately. For the first two events we have

$$\begin{aligned}\Pr[\mathcal{E}_1(e)] &= \left(\frac{1}{2} - \delta(e)\right)^{|e|} \le 2^{-|e|}\exp(-2\delta(e)|e|) \le 2^{-|e|}\exp(-\alpha)\,;\\ \Pr[\mathcal{E}_2(f)] &= \left(\frac{1}{2} - \delta(f)\right)^{|f|} \le 2^{-|f|}\exp(-2\delta(f)|f|) \le 2^{-|f|}\exp(-\alpha).\end{aligned}$$

To bound the probability of the third event, namely $\mathcal{E}_3(e,f)$, note that if both $\neg\mathcal{E}_1(e)$ and $\neg\mathcal{E}_2(f)$ hold, then $\eta(v) \in [\frac{1}{2} - \delta(e), \frac{1}{2} + \delta(f)]$. We condition on $\eta(v) = \frac{1}{2} + x$ and integrate to obtain

$$\begin{aligned}\Pr[\mathcal{E}_3(e,f)] &\le \int_{-\delta(e)}^{\delta(f)} \left(\frac{1}{2} + x\right)^{|e|-1} \left(\frac{1}{2} - x\right)^{|f|-1} \mathrm{d}x\\ &= 2^{-|e|-|f|+2} \int_{-\delta(e)}^{\delta(f)} (1+2x)^{|e|-1}(1-2x)^{|f|-1}\,\mathrm{d}x\\ &\le 2^{-|e|-|f|+2} \cdot \beta(e,f),\end{aligned}$$

where

$$\beta(e,f) = \begin{cases} \int_{-\delta(f)}^{\delta(e)} (1+2x)^{|f|-|e|}\,\mathrm{d}x & \text{if } |e| \le |f|;\\ \int_{-\delta(e)}^{\delta(f)} (1+2x)^{|e|-|f|}\,\mathrm{d}x & \text{if } |e| > |f|.\end{cases} \tag{3}$$

(Both bounds for $\beta(e,f)$ follow from the fact that $(1+2x)\cdot(1-2x) \le 1$.)

We next show the following.

▷ Claim 5. For all large k, we have

$$\beta(e,f) \le \max_{\lambda' \in [0,\lambda]} \frac{1/2}{(\lambda'-1)k+1}\left[\left(1+\frac{\alpha}{k}\right)^{(\lambda'-1)k+1} - \left(1-\frac{\alpha}{\lambda' k}\right)^{(\lambda'-1)k+1}\right] \tag{4}$$

Note that the right hand side of Equation (4) is precisely $\gamma(k,\lambda,\alpha)^{-2}$. Claim 4 then follows from bound (2). ◁

Proof of Claim 5. Suppose $|f| \ge |e|$ (the case $|f| < |e|$ is similar). Set $|f| = \lambda'|e|$; note that $1 \le \lambda' \le \lambda$. Then,

$$\beta(e,f) = \frac{1/2}{(\lambda'-1)|e|+1}\left[\left(1+\frac{\alpha}{|e|}\right)^{(\lambda'-1)|e|+1} - \left(1-\frac{\alpha}{\lambda'|e|}\right)^{(\lambda'-1)|e|+1}\right]$$

Let

$$h_{\lambda'}(x) \doteq \frac{1}{(\lambda'-1)x+1}\left[\left(1+\frac{\alpha}{x}\right)^{(\lambda'-1)x+1} - \left(1-\frac{\alpha}{\lambda' x}\right)^{(\lambda'-1)x+1}\right]$$

To establish our claim, we will show that $h_{\lambda'}(x)$ is a decreasing function for x in the range $[k,\infty)$, so it is maximum for $x=k$. Indeed, the derivative $h'_{\lambda'}(x)$ is the sum of three terms:

$$-\frac{(\lambda'-1)}{((\lambda'-1)x+1)^2}\left[\left(1+\frac{\alpha}{x}\right)^{(\lambda'-1)x+1} - \left(1-\frac{\alpha}{\lambda' x}\right)^{(\lambda'-1)x+1}\right], \tag{5}$$

$$\frac{1}{(\lambda'-1)x+1}\left[\left(1+\frac{\alpha}{x}\right)^{(\lambda'-1)x+1}\left(\ln(1+\frac{\alpha}{x})(\lambda'-1) - ((\lambda'-1)x+1)\frac{\alpha}{x(\alpha+x)}\right)\right], \tag{6}$$

and

$$-\left(\frac{1}{(\lambda'-1)x+1}\right)\left(1-\frac{\alpha}{\lambda' x}\right)^{(\lambda'-1)x+1} \times \left(\ln(1-\frac{\alpha}{\lambda' x})(\lambda'-1) + ((\lambda'-1)x+1)\frac{\alpha}{x(\lambda' x-\alpha)}\right). \tag{7}$$

We wish to show that this derivative is negative for $x \in [k,\infty)$. First, we show that term (7) is negative, but verifying that the three factors in parentheses are positive. The first two factors are clearly positive. We rearrange the last factor as

$$(\lambda'-1)\left(\ln(1-\frac{\alpha}{\lambda' x}) + \frac{\alpha}{\lambda' x-\alpha}\right) + \frac{\alpha}{x(\lambda' x-\alpha)}, \tag{8}$$

and using $\ln(1-\frac{\alpha}{\lambda' x}) = -\ln(1+\frac{\alpha}{\lambda' x-\alpha}) \geq \frac{-\alpha}{\lambda' x-\alpha}$, verify that this factor is positive as well.

We now deal with the first two terms. The first term (5) is at most

$$-\frac{(\lambda'-1)}{((\lambda'-1)x+1)^2}\left(1+\frac{\alpha}{x}\right)^{(\lambda'-1)x+1}\left[1-\left(1-\frac{\alpha}{\lambda' x}\right)^{(\lambda'-1)x+1}\right], \tag{9}$$

Using the inequality $\ln(1+\frac{\alpha}{x}) \leq \frac{\alpha}{x}$, we see that the second term (6) is at most

$$\frac{1}{(\lambda'-1)x+1}\left(1+\frac{\alpha}{x}\right)^{(\lambda'-1)x+1}\left((\lambda'-1)\frac{\alpha^2}{x(x+\alpha)} - \frac{\alpha}{x(x+\alpha)}\right) \tag{10}$$

$$\leq \frac{\lambda'-1}{(\lambda'-1)x+1}\left(1+\frac{\alpha}{x}\right)^{(\lambda'-1)x+1}\left(\frac{\alpha^2}{x(x+\alpha)}\right). \tag{11}$$

Thus, since the third term (7) is negative, the bounds (9) and (11) on the first two terms, (5) and (6), imply that the derivative $h'_{\lambda'}(x)$ is at most $\frac{\lambda'-1}{(\lambda'-1)x+1}\left(1+\frac{\alpha}{x}\right)^{(\lambda'-1)x+1}$ times

$$-\frac{1}{(\lambda'-1)x+1} + \frac{\alpha^2}{x(x+\alpha)} + \frac{1}{(\lambda'-1)x+1}\left(1-\frac{\alpha}{\lambda' x}\right)^{(\lambda'-1)x+1}$$

$$= -\frac{x(x+\alpha)(1-\left(1-\frac{\alpha}{\lambda' x}\right)^{(\lambda'-1)x+1}) - \alpha^2((\lambda'-1)x+1)}{((\lambda'-1)x+1)x(x+\alpha)}. \tag{12}$$

We will verify that the numerator of the fraction in Equation (12) is positive. First note that by the weighted AM-GM inequality $(1-\epsilon)^z(1+\epsilon z) \leq 1^{1+z} = 1$ (for $0 \leq \epsilon < 1$ and $z \geq 0$); thus, $1-(1-\epsilon)^z \geq \epsilon z/(1+\epsilon z)$. In the following let $z \doteq (\lambda'-1)x+1$ and $\epsilon \doteq \alpha/(\lambda' x)$; then, the numerator in Equation (12) is at least

$$\begin{aligned} x(x+\alpha)\frac{\alpha z/(\lambda' x)}{1+\alpha z/(\lambda' x)} - \alpha^2 z &= \frac{\alpha z}{\lambda' x + \alpha z}\left[x(x+\alpha) - \alpha(\lambda' x + \alpha z)\right] \\ &= \frac{\alpha z}{\lambda' x + \alpha z}\left[x(x - \alpha(\lambda' + \alpha\lambda' - \alpha)) + \alpha(x-\alpha)\right] \\ &= \frac{\alpha z}{\lambda' x + \alpha z}\left[x^2 - x(\alpha\lambda + \alpha^2\lambda - \alpha^2) + \alpha(x-\alpha)\right] \\ &\geq \frac{\alpha z}{\lambda' x + \alpha z}\left[x(x - 2\alpha^2\lambda) + x\alpha^2 + \alpha(x-\alpha)\right]. \end{aligned}$$

Our assumption $2\alpha^2\lambda \leq k \leq x$ implies that the last quantity, and hence the numerator of the fraction in Equation (12), is positive. Thus, $h_{\lambda'}(x)$ is decreasing over $[k,\infty)$ and

$$\beta(e,f) \leq \frac{1}{2}\max_{\lambda'\in[1,\lambda]} h_{\lambda'}(k).$$

Claim 5 follows from this. ◁

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A Proof of Lemma 2

Proof. Recall that

$$\gamma(k,\lambda,\alpha)^2 \doteq \min_{\lambda' \in [1,\lambda]} 2((\lambda'-1)k+1)\left[\left(1+\frac{\alpha}{k}\right)^{(\lambda'-1)k+1} - \left(1-\frac{\alpha}{\lambda' k}\right)^{(\lambda'-1)k+1)}\right]^{-1}.$$

We will show that reciprocal of the expression under the minimum is at most $(\frac{\alpha}{k})\exp(\alpha\lambda)$. It will follow that $\gamma(k,\lambda,\alpha)^2 \geq (\frac{k}{\alpha})\exp(-\alpha\lambda)$.

Let $z \doteq (\lambda'-1)k+1$. We will consider two cases, based on whether or not $2z\alpha \geq k$. First, suppose $2z\alpha \geq k$, that is, $z \geq k/(2\alpha)$. Then, we have the desired upper bound

$$\frac{1}{2z}\left[\left(1+\frac{\alpha}{k}\right)^z - \left(1-\frac{\alpha}{\lambda' k}\right)^z\right] \leq \frac{1}{2z}\left(1+\frac{\alpha}{k}\right)^z \leq \left(\frac{\alpha}{k}\right)\exp(\alpha z/k) \leq \left(\frac{\alpha}{k}\right)\exp(\alpha\lambda).$$

Next assume $2z\alpha < k$. We will use the following inequalities for bounding expressions of the form $(1+x)^\ell$. For $\ell \geq 1$ and $\ell|x| < 1$ (note x may be negative), we have $1+\ell x \leq (1+x)^\ell \leq 1/(1-\ell x)$. Then

$$\begin{aligned}
\frac{1}{2z}\left[\left(1+\frac{\alpha}{k}\right)^z - \left(1-\frac{\alpha}{\lambda' k}\right)^z\right] &\leq \frac{1}{2z}\left[\frac{1}{1-z\alpha/k} - \left(1-\frac{z\alpha}{\lambda' k}\right)\right] \quad (\text{note } z\alpha/k < \tfrac{1}{2})\\
&= \frac{1}{2z}\left(\frac{1-(1-z\alpha/k)(1-z\alpha/(\lambda' k))}{1-z\alpha/k}\right)\\
&\leq \frac{\alpha+\alpha/\lambda' - z\alpha^2/(\lambda' k)}{2(k-z\alpha)}\\
&\leq \frac{\alpha+\alpha/\lambda' - z\alpha^2/(\lambda' k)}{k} \quad (\text{since } 2z\alpha \leq k), \lambda' \geq 1)\\
&\leq \frac{2\alpha}{k}\\
&\leq \left(\frac{\alpha}{k}\right)\exp(\alpha\lambda). \quad (\text{since } \alpha,\lambda \geq 1)
\end{aligned}$$

◀