Domatic Partitions and the Lovász Local Lemma

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1 Our result.

We resolve the problem posed as the main open question in [4]: letting $\delta(G)$, $\Delta(G)$ and $D(G)$ respectively denote the minimum degree, maximum degree, and domatic number (defined below) of an undirected graph $G = (V, E)$, we show that $D(G) \geq (1 - o(1))\delta(G)/\ln(\Delta(G))$, where the "$o(1)$" term goes to zero as $\Delta(G) \to \infty$. A *dominating set* of G is any set $S \subseteq V$ such that for all $v \in V$, either $v \in S$ or some neighbor of v is in S. A *domatic partition* of V is a partition of V into dominating sets, and the number of these dominating sets is called the *size* of such a partition. The *domatic number* $D(G)$ of G is the maximum size of a domatic partition; it is NP-hard to find a maximum-sized domatic partition. This is a very well-studied problem especially for various special classes of (perfect) graphs: see, e.g., [2, 6, 7] and the references in [4].

Recent interesting work of [4] has given the first non-trivial approximation algorithm for the domatic partition problem, whose approximation guarantee is also shown to be essentially best-possible in [4]. Let $n = |V|$, $\delta = \delta(G)$, and $\Delta = \Delta(G)$. It is easy to check that $D(G) \leq \delta + 1$. An efficient algorithm to find a domatic partition of size $(1 - o(1))\delta/\ln n$ is shown in [4], where the $o(1)$ term goes to zero as n increases; thus, this is a $(1 + o(1))\ln n$ approximation. It is also shown in [4] that for any fixed $\epsilon > 0$, an $(1 - \epsilon)\ln n$-approximation algorithm for $D(G)$ would imply that $NP \subseteq DTIME[n^{O(\log\log n)}]$; hence such an algorithm appears unlikely. An interesting point is that this seems to be the first natural *maximization* problem proven to have a $\Theta(\log n)$ approximation threshold. Can we say something better for sparse graphs? It is shown in [4] that $D(G) \geq (1 - o(1))\delta/(3\ln\Delta)$, where the $o(1)$ term is a function of Δ that goes to zero as Δ increases. (Among the very few such lower bounds known before was that $D(G) \geq \lceil n/(n - \delta)\rceil$ [8]. This is relevant primarily for very dense graphs. For instance, when $1 \leq \delta \leq n/2$, this bound says that $D(G) \geq 2$; however, $D(G) \geq 2$ is readily seen to hold if (and only if) $\delta \geq 1$. If $\delta \geq 1$, then any maximal independent set and its complement are dominating sets.) One of the main open questions asked in [4] is whether $D(G) \geq (1 - o(1))\delta/\ln\Delta$; we answer this in the affirmative. (As a step in this direction, the problem is resolved in [4] for graphs with girth at least 5.) This $D(G)$ bound is best-possible up to lower-order terms.

We are mainly motivated by the work of [4]; as in [4], the Lovász Local Lemma (LLL) [3] will be key to our analysis. The main difference is that we apply a "slow" (two-stage) partitioning that helps prune the dependencies in applying the LLL, leading to our improvement. The first partition is small-sized, but has properties much stronger than being domatic; the second partition refines the first, and crucially benefits from these useful properties of the first partition.

THEOREM 1.1. *There is a constant $a > 0$ such that for any graph G with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G) \geq 3$, $D(G) \geq \lfloor \frac{\delta}{\ln\Delta + a\ln\ln\Delta} \rfloor$. (The value of $D(G)$ for $\Delta \leq 2$ is known via a case analysis, and there is a corresponding linear-time algorithm.)*

2 Proof sketch for Theorem 1.1.

We will assume throughout that Δ is large enough, i.e., for some constant Δ_0, we assume that $\Delta \geq \Delta_0$. (If $\Delta \leq \Delta_0$, Theorem 1.1 holds by setting a large enough.) Define $[k] \doteq \{1, 2, \ldots, k\}$, and let $\exp(x)$ denote e^x. Let $N(v)$ denote the set of neighbors of vertex v, and define $N^+(v) = \{v\} \cup N(v)$; let $d(v) = |N^+(v)|$. Suppose $\delta \leq \ln^4\Delta$. Letting $\ell = \lfloor \delta/(\ln\Delta + a\ln\ln\Delta)\rfloor$, choose a color uniformly at random from $[\ell]$, independently for each vertex. We can show via appropriate use of the general (asymmetric) version of the LLL (Lemma 1.1 in Chapter 5 of [1]) that with positive probability, all colors will be covered by $N^+(v)$ for each v.

Next consider the more challenging case where $\delta > \ln^4\Delta$. Let $\epsilon = 1/(\ln\Delta)$. Define $\ell_1 = \lfloor \epsilon^3\delta \rfloor$ and $\ell_2 = \lfloor \ln^2\Delta/(1 + b(\ln\ln\Delta)/\ln\Delta)\rfloor$ for a suitable constant b. We will show the existence of a coloring $\chi : V \to [\ell_1] \times [\ell_2]$ such that for every vertex u, there is at least one vertex of each color in $N^+(u)$. We apply a two-stage coloring: the first coloring determines the first components of the vertex-colors, and the second

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coloring is for the second components. Thus, the first coloring is a coarse partition, which the second coloring turns into a fine partition. The crucial role of the first coloring is to reduce certain dependencies in applying the LLL to analyze the second coloring.

The first coloring independently colors each vertex with a random color from $[\ell_1]$. By an application of the Chernoff bounds and the symmetric version of the LLL (Corollary 1.2 in Chapter 5 of [1]), we can show that with positive probability, the number of vertices in $N^+(u)$ with color c lies in the range $d(u)(1 \pm 3\epsilon)/\ell_1$, for all pairs (u, c). (Note that this property is much stronger than the coloring being domatic.) Fix any such "good" coloring $\chi_1 : V \to [\ell_1]$. In the second coloring, choose a random color $\chi_2(u) \in [\ell_2]$ for each u, independently of all other vertices; the final color of u is the pair $(\chi_1(u), \chi_2(u))$. Let $\mathcal{B}_{u,c_1,c_2}$ be the bad event that there is no vertex of color (c_1, c_2) in $N^+(u)$. We use the asymmetric LLL to show that all these bad events can be avoided with positive probability. For each vertex u and each $c \in [\ell_1]$, let $N^+_{u,c} = \{v \in N^+(u) : \chi_1(u) = c\}$. Fix an event $\mathcal{B}_{u,c_1,c_2}$. We can show that $\Pr[\mathcal{B}_{u,c_1,c_2}] \leq \exp(-(1-3\epsilon) \cdot (1 + b(\ln\ln\Delta)/\ln\Delta) \cdot (\ln\Delta) \cdot d(u)/\delta) \leq (\Delta(\ln\Delta)^b)^{-(1-3\epsilon)}$. Which other events does $\mathcal{B}_{u,c_1,c_2}$ depend on? For $S \subseteq V$, let $N^+(S) \doteq \bigcup_{v \in S} N^+(v)$. We can check that $\mathcal{B}_{u,c_1,c_2}$ only depends on the events in $S(u, c_1, c_2) = \{\mathcal{B}_{v,c'_1,c'_2} : v \in N^+(N^+_{u,c_1}) \text{ and } c'_1 = c_1\}$. By the general LLL, it suffices to display a real $y_{u,c_1,c_2} \in (0, 1)$ for each (u, c_1, c_2), such that for all (u, c_1, c_2),

$$\Pr[\mathcal{B}_{u,c_1,c_2}] \leq y_{u,c_1,c_2} \cdot \prod_{(v,c'_1,c'_2) \in S(u,c_1,c_2)} (1 - y_{v,c'_1,c'_2}).$$

This bound can be shown via the properties of the first coloring by choosing $y_{v,c'_1,c'_2} = \exp(-(1-\epsilon) \cdot (1 - 3\epsilon) \cdot (1 + b(\ln\ln\Delta)/\ln\Delta) \cdot (\ln\Delta) \cdot d(v)/\delta)$.

A complete proof of this result appears in [5]. An interesting open problem is to obtain an algorithmic version of this lower bound on the domatic number.

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