CMSC 858S: Randomized Algorithms

Fall 2001

Homework Assignment #1 **(will be graded)**

Due date: Beginning of class on October 5, 2001

Please note: Please submit only a final, polished set of answers, and do not include unnecessary material. (For instance, please don't include your preliminary work/calculations/intuition that led to the final answer.) Partial credit will be given where appropriate: if you think you have some ideas that deserve partial credit, please *itemize them concisely*, so that the grading process can be accurate.

1. Let $G = (V, E)$ be a graph with n vertices and minimum degree $\delta \geq 11$. Show that there is a partition of V into two subsets A and B such that $|A| \leq O(n \log(\delta + 1)/(\delta + 1))$, and such that each vertex of B has at least one neighbor in A and at least one neighbor in B. **(10 points)**

Hint: Start with a simple random process followed by a carefully-designed alteration.

2. Recall that a random variable X has a Poisson distribution with mean μ if X takes on only non-negative integer values, with $\Pr[X = i] = e^{-\mu}\mu^i/i!$ for $i \geq 0$. (Verify for yourself that $\mathbf{E}[X] = \mu$; you don't need to prove this as part of this problem.) Use the method we used in developing the Chernoff bound, to get an upper bound on $\Pr[X \geq \mu(1 + \delta)]$ for $\delta > 0$. Do not express the result as an infinite sum etc.: express it in closed form. Do you see a similarity with the Chernoff bound we derived in class? **(5 points)**

3. Consider the following random process. We have n players and n bins. The process proceeds in rounds as follows. Initially, all the bins are empty, and each player independently throws a ball into one of the bins at random. This is the first round. In general, in round $i + 1$ (where $i \geq 0$), each player throws a new ball at random into one of the bins that was empty at the end of round i. (No ball thrown into a bin is ever removed from the bin.) The process stops when all bins are nonempty. We wish to study the number of rounds taken by this process.

(a) Suppose we are given that exactly $e_i \geq 1$ bins are empty at the end of round i. What is the expected number of empty bins at the end of round $i + 1$? **(2 points)**

(b) In the notation of part (a), suppose $e_i \leq n/(3 \ln n)$ for some i. Show that with high probability (i.e., with probability tending to 1 as $n \to \infty$), the process will stop at the end of round $i + 1$. **(2 points)**

(c) Recall that the function $\log^* x$ is defined, for $x \geq 1$, to be the number of times we need to iterate the logarithm (to the base 2) on x, until the result becomes non-positive. [For instance, $\log^* 1 = 1$, $\log^* x = 2$ for $1 < x \leq 2$, $\log^* x = 3$ for $2 < x \leq 4$, and $\log^* x = 4$ for $4 < x \leq 16$.] Use parts (a) and (b) with additional ideas to show that our random process terminates within $O(\log^* n)$ rounds with high probability. **(6 points)**