1. Suppose a function $f(x)$ satisfies $|f'''(x)| \leq M$ for $x \in [-1,1]$. Let $p(x)$ be the quadratic polynomial interpolating $f(x)$ at the points $-1,0,1$. Give a bound on $|p(x) - f(x)|$ for $x \in [-1,1]$. The better your bound, the better your grade.

Answer: The absolute value error is of the error is

$$ |f(x) - p(x)| \leq \frac{f'''(\eta)}{6} |x(x-1)(x+1)|,$$

where $\eta \in [0,1]$. Hence

$$ |f(x) - p(x)| \leq \frac{M}{6} \max_{t \in [0,1]} |x(x-1)(x+1)|.$$

Now by differentiating $x(x-1)(x+1) = x^3 - x$ and setting the result to zero, we see that $|x(x-1)(x+1)|$ is maximized at $x = \pm \frac{1}{\sqrt{3}}$ and that the maximum is $\frac{2}{3\sqrt{3}}$. Hence our bound is

$$ |f(x) - p(x)| \leq \frac{M}{9\sqrt{3}}.$$

2. We can evaluate the polynomial $p(x) = c_n x^{n-1} + c_{n-1} x^{n-2} + \cdots + c_2 x + c_1$ by the recurrence

$$ p_n = c_n $$

$$ p_i = x p_{i+1} + c_i, \quad i = n-1, \ldots, 1.$$

The value of $p(x)$ is $p_1$.

a. By differentiating the recursion, derive a recursion to evaluate $p'(x)$.

b. Assume that

$$ p(x) = c_n (x-x_{n-1}) \cdots (x-x_1) + c_{n-1} (x-x_{n-2}) \cdots (x-x_1) + \cdots + c_2 (x-x_1) + c_1 $$

is in Newton form. Write down recurrences for evaluating $p(x)$ and $p'(x)$.

Answer:

a. From the recurrence we obtain

$$ p'_n = 0 $$

$$ p'_i = xp'_{i+1} + p_{i+1}, \quad i = n-1, \ldots, 1.$$

This can be simplified to

$$ p'_{n-1} = p_n $$

$$ p'_i = xp'_{i+1} + p_{i+1}, \quad i = n-2, \ldots, 1.$$
b. The recurrence to evaluate $p(x)$ is

$$p_n = c_n$$
$$p_i = (x - x_i)p_{i+1} + c(i), \quad i = n - 1, \ldots, 1.$$  

The recurrence to evaluate $p'(x)$ is

$$p'_n = 0$$
$$p'_i = (x - x_i)p'_{i+1} + p_{i+1}, \quad i = n - 1, \ldots, 1.$$  

As above this recurrence can be simplified.

3. The weights for the barycentric interpolation formula at the points

$$(x_1, f_1), (x_2, f_2), \ldots, (x_n, f_n)$$

are

$$w_i = \frac{1}{\prod_{j \neq i} (x_i - x_j)}.$$  

Suppose we are given an additional point $(x_{n+1}, f_{n+1})$. Show how to compute the weights $\hat{w}_i$ ($i = 1, \ldots, n + 1$) of the new interpolant from the weights $w_i$ in $O(n)$ operations.

Answer:

$$\hat{w}_i = w_i / (x_i - x_{i+1}), \quad i = 1, \ldots, n,$$
$$\hat{w}_{n+1} = 1 / \prod_{i=1}^{n} (x_{n+1} - x_i).$$