

1. Suppose a function $f(x)$ satisfies $|f'''(x)| \leq M$ for $x \in [-1, 1]$. Let $p(x)$ be the quadratic polynomial interpolating $f(x)$ at the points $-1, 0, 1$. Give a bound on $|p(x) - f(x)|$ for $x \in [-1, 1]$. The better your bound, the better your grade.

Answer: The absolute value error is of the error is

$$|f(x) - p(x)| \leq \frac{f'''(\eta)}{6} |x(x-1)(x+1)|,$$

where $\eta \in [0, 1]$. Hence

$$|f(x) - p(x)| \leq \frac{M}{6} \max_{t \in [0, 1]} |x(x-1)(x+1)|.$$

Now by differentiating $x(x-1)(x+1) = x^3 - x$ and setting the result to zero, we see that $|x(x-1)(x+1)|$ is maximized at $x = \pm \frac{1}{\sqrt{3}}$ and that the maximum is $\frac{2}{3\sqrt{3}}$. Hence our bound is

$$|f(x) - p(x)| \leq \frac{M}{9\sqrt{3}}.$$

2. We can evaluate the polynomial $p(x) = c_n x^{n-1} + c_{n-1} x^{n-2} + \dots + c_2 x + c_1$ by the recurrence

$$\begin{aligned} p_n &= c_n \\ p_i &= x p_{i+1} + c_i, \quad i = n-1, \dots, 1. \end{aligned}$$

The value of $p(x)$ is p_1 .

- By differentiating the recursion, derive a recursion to evaluate $p'(x)$.
- Assume that

$$p(x) = c_n(x - x_{n-1}) \cdots (x - x_1) + c_{n-1}(x - x_{n-2}) \cdots (x - x_1) + \cdots + c_2(x - x_1) + c_1$$

is in Newton form. Write down recurrences for evaluating $p(x)$ and $p'(x)$.

Answer:

a. From the recurrence we obtain

$$\begin{aligned} p'_n &= 0 \\ p'_i &= x p'_{i+1} + p_{i+1}, \quad i = n-1, \dots, 1. \end{aligned}$$

This can be simplified to

$$\begin{aligned} p'_{n-1} &= p_n \\ p'_i &= x p'_{i+1} + p_{i+1}, \quad i = n-2, \dots, 1. \end{aligned}$$

b. The recurrence to evaluate $p(x)$ is

$$\begin{aligned} p_n &= c_n \\ p_i &= (x - x_i)p_{i+1} + c(i), \quad i = n - 1, \dots, 1. \end{aligned}$$

The recurrence to evaluate $p'(x)$ is

$$\begin{aligned} p'_n &= 0 \\ p'_i &= (x - x_i)p'_{i+1} + p_{i+1}, \quad i = n - 1, \dots, 1. \end{aligned}$$

As above this recurrence can be simplified.

3. The weights for the barycentric interpolation formula at the points

$$(x_1, f_1), (x_2, f_2), \dots, (x_n, f_n)$$

are

$$w_i = \frac{1}{\prod_{j \neq i} (x_i - x_j)}.$$

Suppose we are given an additional point (x_{n+1}, f_{n+1}) . Show how to compute the weights \hat{w}_i ($i = 1, \dots, n + 1$) of the new interpolant from the weights w_i in $O(n)$ operations.

Answer:

$$\begin{aligned} \hat{w}_i &= w_i / (x_i - x_{i+1}), \quad i = 1, \dots, n, \\ \hat{w}_{i+1} &= 1 / \prod_{i=1}^n (x_{n+1} - x_i). \end{aligned}$$