Show work for all three problems. If you used Matlab, hand in scripts showing what you did. The following are useful fodder for the Matlab’s help command: horzcat, vertcat, mldivide, roots.

1. Use the method of undetermined coefficients to derive the “half simp” formula

\[
\int_{1/2}^{1} f(x) \, dx \approx A_0 f(0) + A_1 f(1/2) + A_2 f(1);
\]

[Hint: Use Matlab to solve the linear system.]

**Answer:** We have the three equations

\[
\begin{align*}
\int_{1/2}^{1} 1 \, dx &= \frac{1}{2} = A_0 + A_1 + A_2 \\
\int_{1/2}^{1} x \, dx &= \frac{3}{8} = \frac{1}{2} A_1 + A_2 \\
\int_{1/2}^{1} x^2 \, dx &= \frac{7}{24} = \frac{1}{4} A_1 + A_2
\end{align*}
\]

the solution is

```matlab
format long
c = [1 1 1; 0, 1/2, 1; 0, 1/4, 1];
b = [1/2; 3/8; 7/24];
a = c\b;
a =
-0.04166666666667
0.33333333333333
0.20833333333333
```

2. We wish to approximate

\[
\int_{0}^{\infty} f(x) e^{-x} \, dx = A_0 f(0) + A_1 f(3) + A_2 f(5).
\]

Use the method of undetermined coefficients to determine the weights \( A_i \). Apply your quadrature rule to approximate

\[
\int_{0}^{\infty} \text{sech} x \, dx = \frac{\pi}{2} = 1.5707 \ldots
\]

[Hints: First find a formula for \( \int_{0}^{\infty} x^n e^{-x} \, dx \). It will then be easy to set up a \( 3 \times 3 \) for the \( A_i \).]

**Answer:** The system is determined as in the first problem. Using the fact that \( \int_{0}^{\infty} x^n = n! \), we get the following Matlab script.

```matlab
1
```
\( \mathbf{C} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 9 & 25 \end{bmatrix} \)
\( \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \)
\( \mathbf{A} = \mathbf{C}\backslash\mathbf{b} \)

\[ \mathbf{A} = \begin{bmatrix} 0.60000000000000 \\ 0.50000000000000 \\ -0.10000000000000 \end{bmatrix} \]

The evaluation of the integral (*) goes as follows.

\( \text{absc} = [0, 3, 5] \)
\( \mathbf{f} = \text{sech}(\text{absc}).*\exp(\text{absc}); \)
\( \mathbf{apprx} = \mathbf{f}\mathbf{A} \)

\[ \mathbf{apprx} = \\ 1.39753645641711 \]

3. Compute the abscissas and weights for the Gauss–Leguerre quadrature formula

\[ \int_{0}^{\infty} \mathbf{x} \) e^{-x} dx = A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2); \]

Use it to approximate the integral (*). Compare your results with those of Problem 2.

[Hints: This involves three steps. First, you must find the coefficients of the cubic Laguerre polynomial \( p(x) = x^3 - ax^2 - bx - c \) satisfying \( \int_{0}^{\infty} x^i p(x) e^{-x} = 0 \) \((i = 0, 1, 2)\). Then you must find the roots of this polynomial. Finally, you must determine the weights as usual. A three line Matlab script will do all of this.]

Answer: Here is the Matlab script that evaluates the abscissas and weights.

\[ \text{coef} = [1;-([2,1,1;6,2,1;24,6,2]\backslash[6;24;120])]] \]
\( \text{absc} = \text{sort}(\text{roots(coef)}) \)
\( \text{wgt} = [1 1 1;... \]
\( \text{absc}(1), \text{absc}(2), \text{absc}(3);... \]
\( \text{absc}(1)^2, \text{absc}(2)^2, \text{absc}(3)^2]\backslash[1;1;2] \]

\[ \text{coef} = \\ 1.00000000000000 \\ -9.00000000000000 \\ 18.00000000000000 \\ -6.00000000000000 \]
\( \text{absc} = \\ 0.41577455678348 \\ 2.29428036027904 \]
wgt =
0.71109300992917
0.27851773356924
0.01038925650159

The evaluation of the integral is as above.

\[
f = \text{sech}(absc) \cdot \exp(absc);
\]

\[
\text{apprx} = f \ast \text{wgt}
\]

\[
\text{apprx} = 1.56301921023833
\]

This is a much better approximation to \(\pi/2 = 1.5707\ldots\)