AMSC/CMSC 460 **Midterm Exam** **3/27/08**

Work any four of the five problems below. Put the numbers of the problems you worked in the upper left hand corner of the cover of your exam book.

1. The central difference scheme for computing $f'(x)$ at a fixed point x is based on the relation

$$f'(x) = D_h(x) - \frac{h^2}{6} f'''(\eta),$$

where

$$D_h(x) = \frac{f(x+h) - f(x-h)}{2h} \qquad (*)$$

and $\eta \in [x-h, x+h]$.

a. Assuming that $|f'''(x)| \le M$, derive a bound $\tau(h)$ for the truncation error $|f'(x) - D_h(x)|$.

b. Assume that when we evaluate $f(x)$ we actually compute $f^*(x) = f(x) + e(x)$, where $|e(x)| \le \epsilon$. Let $D_h^*(x)$ be the result of replacing $f(x)$ by $f^*(x)$ in $(*)$. Derive a bound $\rho(h)$ for $|D_h^*(x) - D_h(x)|$.

c. The function $s(h) = \rho(h) + \tau(h)$ satisfies

$$|f'(x) - D_h^*(x)| \le s(h).$$

Determine the value h_{opt} for which $s(h)$ is smallest.

Answer:

a. $\tau(h) = \frac{h^2}{6} M$.

b. $\rho(h) = \frac{\epsilon}{h}$.

c. Differentiating $s(x)$ with respect to h and setting the result to zero gives $h_{\text{opt}} = \sqrt[3]{3\epsilon/M}$.

2. Let

$$d = x_1 y_1 + x_2 y_2.$$

Suppose we compute d in floating-point arithmetic with rounding unit ϵ_{M}, and assume that addition and multiplication satisfy

$$\text{fl}(a \circ b) = (a \circ b)(1 + \eta), \quad |\eta| \le \epsilon_{\text{M}}, \quad \circ = +, \times$$

If $\tilde{d}$ is the computed d show that

$$\tilde{d} = x_1 \tilde{y}_1 + x_2 \tilde{y}_2 \qquad (*)$$

for some $\tilde{y}_1$ and $\tilde{y}_2$. Give bounds for the relative error in $\tilde{y}_1$ and $\tilde{y}_2$.

Answer: We have

$$\begin{aligned} s_1 &= \mathrm{fl}(x_1y_1) = x_1y_1(1+\eta_1) \\ s_2 &= \mathrm{fl}(x_2y_2) = x_2y_2(1+\eta_2) \\ \tilde{d} &= \mathrm{fl}(s_1+s_2) = (s_1+s_2)(1+\eta_3) \end{aligned}$$

where $|\eta_i| \le \epsilon_{\mathrm{M}}$ $(i = 1, 2, 3)$. It follows that

$$\tilde{d} = x_1y_1(1+\eta_1)(1+\eta_3) + x_2y_2(1+\eta_2)(1+\eta_3)$$

Setting $\tilde{y}_1 = y_1(1+\eta_1)(1+\eta_3)$ and $\tilde{y}_2 = y_2(1+\eta_2)(1+\eta_3)$ we get $(*)$. Since $\tilde{y}_1 = y_1(1+\eta_1+\eta_2+\eta_1\eta_2)$, the relative error in $\tilde{y}_1$ is bounded by $\epsilon_{\mathrm{M}}(2+\epsilon_{\mathrm{M}})$, with the same bound holding for $\tilde{y}_2$.

3. Let (x_1, f_1) and (x_2, f_2) be given with $x_1 \ne x_2$. We desire to determine constants A and λ such that the function $Ae^{\lambda x}$ interpolates f_1 and f_2 at x_1 and x_2. Give formulas for A and λ. When does the interpolating function fail to exist? [Note: This is a nonlinear interpolation problem, requiring you to solve a system of nonlinear equations of order 2. But because the coefficient A appears linearly, you can get rid of it.]

Answer: If we divide $Ae^{\lambda x_2} = f_2$ by $Ae^{\lambda x_1} = f_1$ we get

$$e^{\lambda(x_2-x_1)} = f_2/f_1,$$

from which if follows that

$$\lambda = \frac{\ln(f_2/f_1)}{x_2 - x_1}.$$

and

$$A = f_1e^{-\lambda x_1} = f_2e^{-\lambda x_2}.$$

The interpolation function does not exist when f_1 or f_2 (but not both) are zero or when f_1 and f_2 have opposite signs.

4. Derive an integration rule of the form

$$\int_1^\infty \frac{f(x)}{x^3}\,dx \cong A_1f(1) + A_2f(3)$$

that is exact for all linear polynomials. What would happen if we tried to determine a rule of the form

$$\int_1^\infty \frac{f(x)}{x^3}\,dx \cong A_1f(1) + A_2f(3) + A_3f(5) \qquad (*)$$

that is exact for all quadratic polynomials?

Answer: Using the method of undetermined coefficients, we have

$$\int_1^\infty x^{-3}\,dx = \frac{1}{2} = A_1 + A_2$$

and

$$\int_1^\infty x^{-2}\,dx = 1 = A_1 + 3A_2$$

from which we get $A_1 = A_2 = \frac{1}{4}$.

If we try to make the formula $(*)$ exact for x^2, we end up with the infinite integral $\int_1^\infty x^{-1}\,dx$.

5. A square U of order n is upper triangular if $i > j$ implies $u_{ij} = 0$; i.e., if all the elements of U below the diagonal are zero as in the following diagram ($n = 4$):

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}. \qquad (*)$$

It can be shown that the product $W = UV$ of two upper triangular matrices U and V of order n is upper triangular.

Give Matlab code for an efficient algorithm to compute the product $W = UV$ There are two factors to keep in mind. First it is unnecessary to compute the lower part of W since it is known to be zero. Second, the (i, j)-element of W is a sum of products of the form $u_{ik}v_{kj}$. if w_{ik} or v_{kj} is zero, then there is no need to compute the product and add it to w_{ij}. Your program should have no if statements. [Hint: the easiest solution is to adapt the dot-product algorithm for matrix-matrix multiplication. It will help to write out the product $W = UV$ as in $(*)$.]

Answer:

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for i=1:n
  for j=i:n
     W(i,j) = U(i,i:j)*V(i:j,j);
  end
end
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