

The Baire Theory of Category

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The Baire theory of category, which classifies sets into two distinct categories, is an important topic in the study of metric spaces. Many results in topology arise from category theory; in particular, the Baire categories are related to a topological property. Because the Baire Category Theorem involves nowhere dense sets in a complete metric space, this paper first develops the concepts of *nowhere dense* and *complete metric space*, then explains the Baire Category Theorem.

1 Complete Metric Spaces

For real numbers under the usual metric, the concept of convergence may already be familiar to the reader. The idea of convergence easily extends to other metric spaces. A sequence converges if the distance between each term and some fixed element of the space (known as the limit) approaches zero.

Definition 1.1 *A sequence $\langle x_n \rangle$ from a metric space (X, d) **converges** to the point $x \in X$ if, given $\varepsilon > 0$, there exists N such that $d(x, x_n) < \varepsilon$ for*

all $n \geq N$. Equivalently, $\langle x_n \rangle$ converges to x if every open ball centered at x contains all but finitely many terms of the sequence [3].

Cauchy sequences are similar to convergent sequences, except that the terms of the sequence, rather than becoming close to a limit, become arbitrarily close to each other.

Definition 1.2 A sequence $\langle x_n \rangle$ from a metric space (X, d) is called a **Cauchy sequence** if, given $\varepsilon > 0$, there exists N such that for all $m > N$ and $n > N$, $d(x_m, x_n) < \varepsilon$ [3].

Proposition 1.3 If the sequence $\langle x_n \rangle$ converges, then it is a Cauchy sequence [3].

Proof: Pick $\varepsilon > 0$. Since $\langle x_n \rangle$ converges, there exists N such that $d(x, x_n) < \varepsilon/2$ for all $n \geq N$. Hence, for $m > N$ and $n > N$, $d(x, x_m) + d(x, x_n) < \varepsilon$. By the triangle inequality, $d(x_m, x_n) \leq d(x, x_m) + d(x, x_n)$, so $d(x_m, x_n) < \varepsilon$ [3]. **QED.**

The converse of Proposition 1.3 is not necessarily true [3]. In the space $(0, 1]$ with the usual metric, the sequence $\langle 1/n \mid n = 1, 2, 3, \dots \rangle$ is a Cauchy sequence. For any ε , $d(x_m, x_n) < \varepsilon$ whenever $m > 1/\varepsilon$ and $n > 1/\varepsilon$. However, this sequence does not converge within the space [1].

Definition 1.4 A **complete metric space** is a metric space in which every Cauchy sequence converges [3].

2 Nowhere Dense Sets

Definition 2.1 A subset A of a topological space X is **nowhere dense** if $\text{Int}(\text{Cl}(A)) = \emptyset$ [4].

Recall that a set A in a space X is *dense* if $\text{Cl}(A) = X$. The following proposition shows how dense and nowhere dense sets are related.

Proposition 2.2 A subset A of a topological space X is nowhere dense if and only if $X - \text{Cl}(A)$ is dense [4].

Proof:

$$\begin{aligned} A \text{ is nowhere dense} &\Leftrightarrow \text{Int}(\text{Cl}(A)) = \emptyset \\ &\Leftrightarrow \text{Ext}(X - \text{Cl}(A)) = \emptyset \\ &\Leftrightarrow \text{Int}(X - \text{Cl}(A)) \cup \text{Bd}(X - \text{Cl}(A)) = X \\ &\Leftrightarrow \text{Cl}(X - \text{Cl}(A)) = X \\ &\Leftrightarrow X - \text{Cl}(A) \text{ is dense. } \mathbf{QED.} \end{aligned}$$

3 The Baire Theory of Category

The following theorem, due to Baire, leads to Baire's theory of category.

Theorem 3.1 Let X be a complete metric space and $\{O_n\}_{n=1}^{\infty}$ a countable collection of dense open subsets. Then $\bigcap_{n=1}^{\infty} O_n$ is dense [3].

Proof: Let $U \subset X$ be open. Because O_1 is dense, i.e. $\text{Cl}(O_1) = X$, $O_1 \cap U \neq \emptyset$. Let $x_1 \in O_1 \cap U$, and let $S_1 \subset O_1 \cap U$, where S_1 is an open

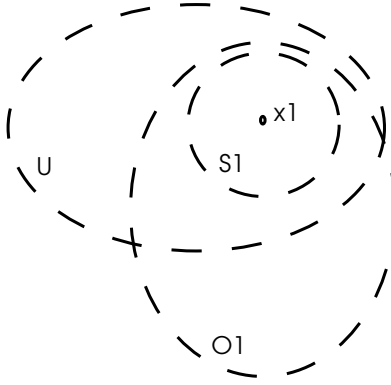


Figure 1: Sets and points used in the proof of Baire's Theorem.

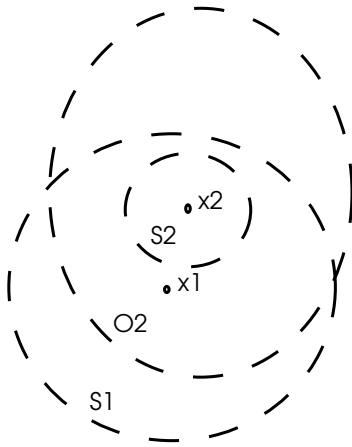


Figure 2: Sets and points used in the proof of Baire's Theorem.

ball centered at x_1 with radius r_1 . (See Figure 1.) Because O_2 is dense, $O_2 \cap S_1 \neq \emptyset$; let $x_2 \in O_2 \cap S_1$. Now, let $S_2 \subset O_2$ be an open ball centered at x_2 with radius $r_2 < \min\{r_1/2, r_1 - d(x_1, x_2)\}$ so that S_2 fits inside of S_1 . (See Figure 2.)

We claim that $Cl(S_2) \subset S_1$. Let C be a closed ball centered at x_2 with radius r_2 . Because r_2 is small enough for S_1 to contain S_2 with room to spare, $Cl(S_2) \subset C \subset S_1$.

If we continue inductively to define S_3, S_4 , and so on, we create a sequence $\langle S_n \rangle$ of balls such that $Cl(S_n) \subset S_{n-1}$, $S_n \subset O_n$, and $\lim_{n \rightarrow \infty} r_n = 0$. We claim that $\langle x_n \rangle$ is a Cauchy sequence. Given ε , we can pick N such that $r_N = \varepsilon/2$. For $m > N$ and $n > N$, $x_m \in S_N$ and $x_n \in S_N$ since $S_n \subset S_{n-1}$. Because x_m and x_n are both inside a ball of radius r_N , $d(x_m, x_n) < 2r_N = \varepsilon$. Thus, $\langle x_n \rangle$ is a Cauchy sequence. Furthermore, since X is complete, $\langle x_n \rangle$ converges to some point in X ; call this point x .

We make a final claim that $x \in (\bigcap_{n=1}^{\infty} O_n) \cap U$. If true, this claim shows that $(\bigcap_{n=1}^{\infty} O_n) \cap U \neq \emptyset$ for an arbitrary open set $U \subset X$, thus $Cl(\bigcap_{n=1}^{\infty} O_n) = X$ and $\bigcap_{n=1}^{\infty} O_n$ is dense. For all $j \geq i$, $x_j \in Cl(S_{i+1}) \subset S_i \subset O_i$. Suppose that $x \in X - Cl(S_{i+1})$. Then $x \in B \subset X - Cl(S_{i+1})$ for some open ball B centered at x . But B is an open ball around x which does not contain x_j for all $j \geq i$. This contradicts the fact that $\langle x_n \rangle$ converges to x , so $x \in Cl(S_{i+1}) \subset S_i \subset O_i$ and hence $x \in (\bigcap_{n=1}^{\infty} O_n)$. $x \in S_1$ implies that $x \in U$ [3]. **QED.**

Other wordings of Baire's Theorem exist. In place of a collection $\{O_n\}_{i=1}^{\infty}$ of dense open sets, Sutherland [4] states the theorem in terms of a collection $H = \{H_n\}_{i=1}^{\infty}$ of nowhere dense sets. In Theorem 3.1, the complement of the collection of dense open sets, $X - (\bigcap_{n=1}^{\infty} O_n) = \bigcup_{n=1}^{\infty} (X - O_n)$, is equivalent to a union of nowhere dense sets. ($X - O_n$ is nowhere dense because $X - Cl(X - O_n) = X - (X - O_n) = O_n$ is dense.) For each n , let $H_n = X - O_n$. Since $\bigcap_{n=1}^{\infty} O_n$ is dense and $\bigcap_{n=1}^{\infty} O_n = X - \bigcup_{n=1}^{\infty} H_n$, $X - H$ is also dense. As Sutherland [4] explains and as this version of the theorem more explicitly shows, "Baire's Theorem says that a countable collection of nowhere dense subsets is not enough fill up a complete metric space."

Using the definitions that follow, a simple rewording of Theorem 3.1 prescribes Baire's system of categories.

Definition 3.2 *A set A is **first category** if A is the union of a countable collection of nowhere dense sets. If A is not first category, then it is **second category** [3].*

Corollary (Baire Category Theorem) 3.3 *Let X be a complete metric space. Then no nonempty open subset of X is of first category [3].*

Proof: Let H be a collection of nowhere dense $U \subset X$ be open. Because $X - H$ is dense, $(X - H) \cap U \neq \emptyset$. Therefore, $U \not\subseteq H$ and H cannot be open [3]. **QED.**

Christenson and Voxman [2] state Baire's Category Theorem slightly differently: "Any complete metric space is of the second category in itself." That a complete metric space X is second category follows directly from Corollary 3.3 because X is open.

Proposition 3.4 *The absence of nonempty open subsets of the first category within a complete metric space is a topological property [3].*

Proof: We must show that, if a complete metric space (X, \mathcal{T}_X) has no nonempty open subsets of first category and if (X, \mathcal{T}_X) is homeomorphic to (Y, \mathcal{T}_Y) , then (Y, \mathcal{T}_Y) also has no nonempty open subsets of first category. Let $f : X \rightarrow Y$ be the homeomorphism. Assume that (Y, \mathcal{T}_Y) has a first category open subset $H = \cup_{n=1}^{\infty} H_n$. Since H is open and f is continuous, $f^{-1}(H)$ is open. Also, $f^{-1}(H) = f^{-1}(\cup_{n=1}^{\infty} H_n) = \cup_{n=1}^{\infty} f^{-1}(H_n)$. $Int(Cl(f^{-1}(H_n)))$ is the union of all open sets that are contained in $Cl(f^{-1}(H_n))$. For an open

set $U \subset Cl(f^{-1}(H_n))$, $f(U) \subset Cl(H_n)$. But, since H_n is nowhere dense, no such U exists, which means that $f^{-1}(H_n)$ is nowhere dense. This contradicts the fact that (X, \mathcal{T}_X) has no first category open subsets. Therefore, (Y, \mathcal{T}_Y) must not have any first category open subsets. **QED.**

References

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