① Introduction & Induction

Welcome to CMSC 351 (Introduction to Algorithms)

Full "Syllabus" is on the website of the course

Prerequisite for the couser: (see the course catalog)

CMSC 250 ← Math 141
CMSC 212 ← CMSC 132

We have homeworks, 2 midterm exams and final exam. One bonus programming project as well in addition of 5% bonus in homeworks as well.

office hours and e-mail contacts of Vahid and me are on the website.

Read the full syllabus very carefully.

In this class you learn what is an algorithm? you learn some types of algorithms (and not all; you may take CS 451 to have a better understanding

We also learn running times of algorithms in this class.

Algorithm comes from the name of Al-Khwarizmi, Muhammad (a persian scientist, mathematician, and author, who developed the concept of the algorithm in mathematics.

An algorithm is an effictive method for solving a problem expressed as a finite sequence of steps. Each algorithm is a list of well-defined instructions for completing a task. starting from an initial state, the instructions describe a computation that proceeds through a well-defined series of successive states, eventually terminating in a final ending state.

Mathematical Induction (as you learned in 250) is a very powerful proof technique which plays a major role in algorithm design. In this session we briefly review induction through examples.

It usually works as follows: let T be a theorem (statement that we want to prove). Suppose T includes a parameter n whose value can be any natural number (a positive integer). Instead of proving directly that T holds for all values of n, we prove the following two conditions:

1. T holds for $n=1$ (Basis of the induction)
2. For every $n>1$, $\underbrace{\text{if T holds for } n-1}_{\text{Induction hypothesis}}$, then T holds for n.

The reason that these two conditions are sufficient is clear. Cond. 1&2 imply directly that T holds for $n=2$. If T holds for $n=2$, then condition 2 implies that T holds for $n=3$ and so on. The induction principle itself is so basic that we consider it as an axiom and it is usually not proved. Proving induction hypothesis is easier in many cases than the proving the thm directly since it comes for free.

Thus, the induction principle is defined as follows:

☆ If a statement P, with a parameter n, is true for $n=1$, and if for every $n>1$, the truth of P for $n-1$ implies its truth for n, then P is true for all natural numbers (sometime we use n and $n+1$ instead of $n-1$ and n).

It is called strong Induction if we use the truth of P for all natural numbers $<n$ instead of

Three simple examples (read and excersice more from chap 2)

Example 1: For all natural numbers x and n, x^n-1 is divisible by $x-1$.

The proof is by induction on n. The theorem is trivially true for $n=1$. Assum $x^{n-1}-1$ is divisible by $x-1$ for all natural numbers x. We prove that x^n-1 is divisible by $x-1$. (the idea is to try to write the expression x^n-1 using $x^{n-1}-1$, i.e.

$$x^n-1 = x\underbrace{(x^{n-1}-1)}_{\text{is divisible by induction hypo.}} + \underbrace{(x-1)}_{\text{trivial}}$$

Example 2: The sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

For $n=1$: $1 = \frac{1(2)}{2} = 1$.

~~Define $S(n) = n(n+1)$~~

Assum the sum of the first n natural number is $S(n) = \frac{n(n+1)}{2}$ Then.

$$S(n+1) = S(n) + n+1 \underset{\text{by Induction Hypo.}}{=} \frac{n(n+1)}{2} + n+1 = \frac{(n+1)(n+2)}{2},$$ which is exactly what we wanted to prove

Example 3: If n is a natural number and $1+x \geq 0$ then $(1+x)^n \geq 1+nx$.

For $n=1$ both sides are equal to $1+x$.

③ For $n+1$ assuming Induction Hypo. for n.

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq \underset{\text{and } 1+x\geq 0}{(\text{by induction})} (1+x)(1+nx) = 1+(n+1)x+\underbrace{nx^2}_{\text{positive}} \geq 1+(n+1)x$$

More deeper examples:

Example 4 A set of lines in the plane is said to be in general position if no two lines are parallel and no three lines intersect at a common point.

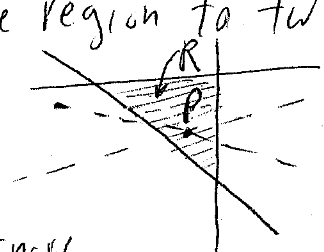
(i.e. no $=$ OR $\ast$).

Thm: The number of regions in the plane formed by n lines in general position $P(n)$ is $\frac{n(n+1)}{2}+1$.

How do we obtain this number? good hint for the right guess can be obtained from small examples: for $n=1$, ther are 2. for $n=2$, there ~~are~~ 4, for $n=3$, there ~~are~~ 7. It seems adding one more line to $n-1$ lines in general position in the plane increases the number of regions by n. If this guess is correct, then the rest is simple (similar to $S(n)$) that we saw before. For $n=1$, $P(1)=2$, for $n+1$, $P(n+1)=P(n)+\underbrace{n+1}_{\text{because of the guess}}=1+\frac{n(n+1)}{2}+n+1$ (because of induction) $=\frac{(n+1)(n+2)}{2}+1$, as desired.

we can prove also the guess itself by induction. For $n=1$, adding one line adds one region (from one region to two). [You can see the rest in the book]

consider the $n+1$ line.



lets remove nth line for know, then we have n line. Thus we have n new regions by adding line $n+1$ to set of lines $1 \ldots n-1$. Now lets pull the nth line back. Since all lines are in general position, the nth and $(n+1)$th line intersects at a point p which must be inside a region R. Both lines thus intersect R. So the addition of $(n+1)$th line, when the nth line is not present cuts R into 2 while when the nth line is present affects R by adding two more} instead of just add one. Furthermore R is the only region affeted so far. Since the two line meet at only one point. Hence the $n+1$ th line adds n regions without the presence of the nth line but it adds $n+1$ regions with the presence of the nth line which completes the proof ▨

Note that here we used two inductions one inside the others.

Another Simple Coloring Problem (Design of Algorithm by Induction)

<u>Example 5</u>

Consider n distinct lines in the plane not necessarily in general position. We want to assign colors to the regions formed by these lines such that neighboring regions (iff they have an edge in common) have different colors. Such an assignment of colors is called a valid coloring.

Thm: It is possible to color the regions formed by any number of lines in the plan with only two colors.

For $n=1$ it is trivial. Assum it is correct for $n-1$. consider the nth line. The only question is how to modify coloring when the nth line is added.

<u>The Algorithm</u>: Divide the regions into two groups according to which side of the nth line they lie. leave all regions on one side colored the same as before, and reverse the colors of all regions on the other side.

<u>To prove that</u> it works consider two neighboring regions R_1 and R_2 and edge e between the

If both are on the sam side of the nth line, then they were colored differently before the line was added (by the induction hypothesis) though they may have the reverse colors, but they are still different.

If the edge between them is part of the nth line, then they belonged to the sam region before the line was added. Since the color of one region was reversed, they are now colored differently.

See more examples in the book.