Improved Semidefinite Programming Hierarchy for Entanglement Testing
with tools from Algebraic Geometry

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Entanglement Detection

Definition (Separable and Entangled States)

A bi-partitie state $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ is *separable* if $\exists$ dist. $\{p_i\}$,

$$\rho = \sum p_i \sigma_X^i \otimes \sigma_Y^i, \text{ s.t. } \sigma_X^i \in \mathrm{D}(\mathcal{X}), \sigma_Y^i \in \mathrm{D}(\mathcal{Y}).$$

Otherwise, ρ is *entangled*. Let Sep $\stackrel{\text{def}}{=}$ { separable states }.

Definition (Entanglement Detection)

A KEY problem: given the description of $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$, decide

Either $\rho \in$ Sep, or ρ is far away from Sep.

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Alternative Formulation

Definition (Weak Membership)

WMem($\epsilon, \|\cdot\|$) : for any $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$, decide either $\rho \in$ Sep or $\|\rho - \text{Sep}\| \geq \epsilon$.

Via standard techniques in convex optimization, equivalent to

Definition (Weak Optimization)

WOpt(M, ϵ) : for any $M \in \mathrm{Herm}(\mathcal{X} \otimes \mathcal{Y})$, estimate the value of

$$\max_{\rho \in \text{Sep}} \langle M, \rho \rangle ,$$

with additive error ϵ.

From now on, we focus on WOpt(M, ϵ).

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Connections

Quantum Information:

- Ground energy that is achieved by *non-entangled* states.
- *Mean-field* approximation in statistical quantum mechanics.
- *Positivity* test of quantum channels.
- *17 more examples* in quantum information in [HM10].

Quantum Complexity:

- Quantum Merlin-Arthur Game with Two-Provers (QMA(2)).

Classical Complexity:

- Unique Game Conjecture and Small-set Expansion. ($\ell_2 \to \ell_4$ norm)

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Early Attempts

Separability Criterions:

- Positive Partial Transpose (PPT) : $\rho^{T_{\mathcal{Y}}} = \rho$? [PH]
- Reduction Criterions: $I_{\mathcal{X}} \otimes \rho_Y \geq \rho$? [HH]
- ......
- FAILURE: any such test has arbitrarily large error. [BS]

Doherty-Parrilo-Spedalieri (DPS) hierarchy:

- ρ is k-extendible if $\exists$ *symmetric* $\sigma \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_k)$, $\forall i, \rho = \sigma_{XY_i}$.
- $\rho \in$ Sep if and only if ρ is k-extendible for any $k > 0$.
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Hardness

Let $h_{\mathrm{Sep}(n)}(M)$ denote the value of

$$\max \langle \mathbf{M}, \rho \rangle \text{ s.t. } \rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y}) \text{ is } \textit{separable},$$

where n refers to the dimension of $\mathcal{X} \otimes \mathcal{Y}$.

Hardness

- **NP**-hard to approximate $h_{\mathrm{Sep}(n)}(M)$ with additive error $\epsilon = 1/poly(n)$. [Gur03,Ioa07,Gha10], [deK08, LQNY09].
- Assuming **Exponential Time Hypothesis (ETH)**, for constant ϵ, approximate $h_{\mathrm{Sep}(n)}(M)$ needs $n^{\Omega(\log(n))}$ time, *via the connection to QMA(2)*. [HM, AB+]

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Upper bounds

When $\epsilon = 1/poly(n)$

- DPS to $O(n/\sqrt{\epsilon})$ level: time $(n/\sqrt{\epsilon})^{O(n)} \rightarrow n^{O(n)}$. [NOP]
- Epsilon-net (brute-force): time $(1/\epsilon)^{O(n)} \rightarrow n^{O(n)}$.

When $\epsilon = \mathrm{const}$

- DPS to $O(log(n)/\epsilon^2)$ level for 1-LOCC M: time $n^{O(log(n)/\epsilon^2)} \rightarrow n^{O(log(n))}$. [BYC, BH]
- Epsilon-net for 1-LOCC M or M with small $\|M\|_2$: time similar to above. [SW, BH]

REMARK: all DPS results correspond to variants of quantum de Finetti theorem.

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Table: Known results about approximating $h_{\mathrm{Sep}(n)}$ to error ϵ

Error ϵ	Lower bounds	Upper b. (DPS)	Upper b. (ϵ-net)
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const	$n^{O(log(n))}$ (ETH)	$n^{O(log(n)/\epsilon^2)}$ (1-LOCC)	similar to left (1-LOCC)

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Error dependence could be SIGNIFICANT

Complexity could grow with $1/\epsilon$

- **Infinite translationally invariant Hamiltonian**: the complexity grows rapidly with $1/\epsilon$ even with fixed local dimension. [CPW]
- **Quantum Interactive Proof**: the complexity jumps from **PSPACE** to **EXP** with smaller ϵ. [IKW]

Will approximating $h_{\mathrm{Sep}(n)}$ be such a case?

REMARK: It is not clear how to improve the error dependence for either DPS or epsilon-net approach.

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Main Result

Error dependence about $h_{\mathrm{Sep}(n)}$

- NO error dependence except *numerical* errors.
- For analytical purposes, there is *no error* at all.
- Numerically, the dependence is $\mathrm{polylog}(1/\epsilon)$, *exponential* improvement from best known $\mathrm{poly}(1/\epsilon)$, $\exp(1/\epsilon)$.

Moreover, the dependence on n **remains the same.**

Theorem (Main)

There exist two algorithms that estimate $h_{\mathrm{Sep}(n)}(M)$ to error ϵ in time $\exp(\mathrm{poly}(n)) \,\mathrm{poly}\log(1/\epsilon)$. similar for the multi-partite case.

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Two Algorithms

Quantifier Elimination

- Based on a generic quantifier elimination solver, to solve

$$\forall W, [\forall |\psi\rangle, |\phi\rangle, \langle\psi| \langle\phi| W |\psi\rangle |\phi\rangle \geq 0 \implies \langle\rho, W\rangle \geq 0].$$

- No new insights into the problem. *Omitted in this talk.*

Improved DPS : DPS+

- Based on DPS hierarchy, with new constraints from **Karush-Kuhn-Tucker** Conditions.
- Formulated as SDPs of similar sizes in terms of the level k.
- The new hierarchy is **exact** when $k = \exp(\mathrm{poly}(n))$.

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$$\forall W, [\forall \left|\psi\right\rangle, \left|\phi\right\rangle, \left\langle\psi\right| \left\langle\phi\right| W \left|\psi\right\rangle \left|\phi\right\rangle \geq 0 \implies \langle \rho, W\rangle \geq 0].$$

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- Based on a generic quantifier elimination solver, to solve

$$\forall W, [\forall |\psi\rangle, |\phi\rangle, \langle\psi| \langle\phi| W |\psi\rangle |\phi\rangle \geq 0 \implies \langle\rho, W\rangle \geq 0].$$

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DPS+ hierarchy

DPS+ hierarchy level k for $h_{\mathrm{Sep}(n)}(M)$

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\max_{\rho} \quad & \langle \rho_{X\mathcal{Y}_1}, M \rangle \\
\text{such that} \quad & \rho \in \mathrm{D}\left(\mathcal{X} \otimes \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_k\right), \\
& \rho \text{ is symmetric on } \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_k, \\
& \langle \rho, \Gamma_i \rangle = 0, \forall i. \quad \text{KKT conditions}
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Remarks

- KKT conditions Γ_i depend on M.
- KKT conditions are written without *multipliers*.

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DPS+ hierarchy as a SDP

- **Primal of SDP**: lead to a new type of monogamy relations. In the eye of any observable M, if the system satisfies **DPS+**, it has no difference from a separable state.
- **Dual of SDP**: lead to a new type of entanglement witness. Similar to [DPS], however, the set of entanglement witness could be *non-convex*.
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- Observe the connection between the DPS hierarchy and the Sum-of-Squares Lasserre/Parrilo hierarchy.
- KKT conditions are necessary for *critical* points.
- KKT conditions imply **finite convergence** (tri-exponential or higher) for a **generic** optimization problem. [N, NR]
- Bring down the level for our problem to exponential.
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The Problem: alternative formulation

Recall that $h_{\mathrm{Sep}(n)}(M)$ refers to

$$\max \langle \mathbf{M}, \rho \rangle \text{ s.t. } \rho \in \mathrm{Sep}(\mathcal{X} \otimes \mathcal{Y}).$$

For any $M \in \mathbb{C}^{n \times n}$, there exists $M' \in \mathbb{C}^{2n \times 2n}$ s.t.

$$h_{\mathrm{ProdSym}(2n)}(M') = \frac{1}{4} h_{\mathrm{Sep}(n)}(M),$$

where $\mathrm{ProdSym}(n,k) := \mathrm{conv}\{(|\psi\rangle\langle\psi|)^{\otimes 2} : |\psi\rangle \in B(\mathbb{C}^n)\}$. [HM]

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Reduce $h_{\mathrm{ProdSym}(n)}$ further

Let $|\psi\rangle = \sum_{i=1}^{n} a_i |i\rangle$ such that $\forall i, a_i \in \mathbb{C}$ and $\sum_i |a_i|^2 = 1$. It is easy to see that $h_{\mathrm{ProdSym}(n)}$ is equivalent to

$$\begin{aligned} \max_{a \in \mathbb{C}^n} \quad & \sum_{i_1, i_2, j_1, j_2} M_{(i_1,i_2),(j_1,j_2)} a_{i_1}^* a_{i_2}^* a_{j_1} a_{j_2} \\ \text{subject to} \quad & \|a\|^2 = 1. \end{aligned} \tag{2}$$

Now reduce from $\mathbb{C}$ to $\mathbb{R}$ by observing:

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$h_{\mathrm{ProdSym}(n)}$ with real variables

By renaming, we arrive at the $h_{\mathrm{ProdSym}(n)}$ with real variables:

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & f_0(x) = \sum_{i_1, i_2, j_1, j_2} M_{(i_1, i_2),(j_1, j_2)} x_{i_1} x_{i_2} x_{j_1} x_{j_2} \\ \text{subject to} \quad & f_1(x) = ||x||^2 - 1 = 0. \end{aligned} \tag{3}$$

REMARK: this is an instance of *polynomial optimization* problems with a homogenous degree 4 objective polynomial and a degree 2 constraint polynomial.

Principle of Sum-of-Squares

One way to show that a polynomial $f(x)$ is *nonnegative* could be

$$f(x) = \sum a_i(x)^2 \geq 0.$$

Example

$$\begin{aligned} f(x) &= 2x^2 - 6x + 5 \\ &= (x^2 - 2x + 1) + (x^2 - 4x + 4) \\ &= (x-1)^2 + (x-2)^2 \geq 0. \end{aligned}$$

Such a decomposition is called a *sum of squares (SOS) certificate* for the non-negativity of f.

Principle of SoS : constrained domain

Definition (Variety)

A set $V \subseteq \mathbb{C}^n$ is called an *algebraic variety* if $V = \{x \in \mathbb{C}^n : g_1(x) = \cdots = g_k(x) = 0\}$.

Non-negativity of $f(x)$ on V could be shown by

$$f(x) = \sum a_i(x)^2 + \sum b_j(x)g_j(x) \geq 0.$$

Question: whether all nonnegative polynomials on certain variety have a SOS certificate? Hilbert 17th problem!

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Putinar's Positivstellensatz

Definition (Ideal)

The *polynomial ideal* I generated by $g_1, \ldots, g_k \in \mathbb{C}[x_1, \ldots, x_n]$ is

$$I = \{\sum a_i g_i : a_i \in \mathbb{C}[x_1, \ldots, x_n]\} =< g_1, \cdots, g_k > .$$

Theorem (Putinar's Positivstellensatz)

Under the Archimedean condition, if $f(x) > 0$ on $V(I) \cap \mathbb{R}^n$, then

$$f(x) = \sigma(x) + g(x),$$

where $\sigma(x)$ is a SOS and $g(x) \in I$.

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SoS in Optimization

$$\begin{aligned} \max \quad & f(x) \\ \text{subject to} \quad & g_i(x) = 0 \quad \forall i \end{aligned} \tag{4}$$

is equivalent to (under AC)

$$\begin{aligned} \min \quad & \nu \\ \text{such that} \quad & \nu - f(x) = \sigma(x) + \sum_i b_i(x) g_i(x), \end{aligned} \tag{5}$$

where $\sigma(x)$ is SOS and $b_i(x)$ is any polynomial.

SoS relaxation: Lasserre/Parrilo Hierarchy

- If $\sigma(x)$ and $b_i(x)$ can have *arbitrarily high* degrees, then the optimization problem (5) is equivalent to problem (4).
- By bounding the degrees, i.e., $\deg(\sigma(x))$, $\deg(b_i(x)g_i(x)) \leq 2D$ for some integer D, we get a hierarchy, namely the Lasserre/Parrilo hierarchy.

$$\begin{aligned} &\min && \nu \\ &\text{such that} && \nu - f(x) = \sigma(x) + \sum_i b_i(x)g_i(x), \end{aligned} \tag{6}$$

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Why it is a SDP?

Observation

- Any $p(x)$ (of degree $2D$) $= m^T Qm$, where m is the vector of monomials of degree up to $2D$ and Q is the coefficients.
- $p(x)$ is a SOS iff $Q \geq 0$.

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Complexity: $\mathrm{poly}(m) \,\mathrm{poly}\log(1/\epsilon)$, where $m = \binom{n+D}{D}$.

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Dual of the SDP: moment

Dual of the SOS cone

- Let $\Sigma_{n,2d}$ be the cone of all PSD matrices representing SOS polynomials with degree up to $2d$.
- The dual cone $\Sigma^*_{n,2d}$ is moment $M_d(x) \geq 0$, where entry (α, β) of $M_d(x)$ is $\int x^{\alpha+\beta}\mu(dx), |\alpha|, |\beta| \leq d$.

Example

When $n = 2, d = 2$, the $M_d(x)$ for homogenous degree 4 moments is given by

$$M_2(x) = \begin{pmatrix} x_{40} & x_{31} & x_{22} \\ x_{31} & x_{22} & x_{13} \\ x_{22} & x_{13} & x_{04} \end{pmatrix} \geq 0$$

Full Symmetry $\Longrightarrow$ DPS

Allow *redundancy*, we can put DPS in this picture.

Example

Now each entry is labelled with $((i,j),(k,l))$ for degree 4 case, i.e., $M_d(x) = \rho \in \mathrm{D}(\mathbb{C}^n \otimes \mathbb{C}^n)$.

$$\rho = \sum_{(i,j),(k,l)} x_i x_j x_k x_l \left|i\right\rangle \left|j\right\rangle \left\langle k\right| \left\langle l\right| .$$

Note that entry $((i,j),(k,l))$ and $((i,l),(k,j))$ have the same value $x_i x_j x_k x_l$. This is **PPT** condition. Similar for **DPS**.

Remark: more symmetry because in ProdSym. Flexible in choosing more or less symmetry.

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Karush-Kuhn-Tucker Conditions

For any optimization problem

$$\max f(x) \text{ s.t. } g_i(x) \leq 0, h_j(x) = 0, \forall i, j,$$

if x^* is a *local* optimizer, then $\exists \mu_i, \lambda_j$,

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Remark: for convex optimization (our case), any global optimizer satisfies KKT.

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Our case

Recall our optimization problem is

$$\max f_0(x) \text{ s.t. } f_1(x) = 0.$$

The KKT condition is $\nabla f_0(x) = \lambda \nabla f_1(x)$, which is equivalent to

$$\operatorname{rank} \begin{pmatrix} \frac{\partial f_0(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_1} \\ \vdots & \vdots \\ \frac{\partial f_0(x)}{\partial x_{2n}} & \frac{\partial f_1(x)}{\partial x_{2n}} \end{pmatrix} < 2.$$

$$g_{ij}(x) = \frac{\partial f_0(x)}{\partial x_i}\frac{\partial f_1(x)}{\partial x_j} - \frac{\partial f_0(x)}{\partial x_j}\frac{\partial f_1(x)}{\partial x_i}, \quad \forall i, j$$

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Optimization Problem with KKT constraints

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\begin{array}{lll}
\text{min} & \nu & \\
\text{such that} & \nu - f_0(x) \geq 0 & \\
& f_1(x) = 0 & \\
\text{KKT} & g_{ij}(x) = 0 & \forall\, 1 \leq i \neq j \leq 2n
\end{array}
$$

- Apply the degree bound D, we get the SoS SDP hierarchy.
- Will show finite convergence when $D = \exp(\text{poly}(n))$. Then $m = \binom{n+D}{D} = \exp(\text{poly}(n))$. Thus the final time is $\exp(\text{poly}(n))\,\text{poly}\log(1/\epsilon)$.

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\text{such that} & \nu - f_0(x) \geq 0 & \\
& f_1(x) = 0 & \\
\text{KKT} & g_{ij}(x) = 0 & \forall\, 1 \leq i \neq j \leq 2n
\end{array}
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- Apply the degree bound D, we get the SoS SDP hierarchy.
- Will show finite convergence when $D = \exp(\text{poly}(n))$. Then $m = \binom{n+D}{D} = \exp(\text{poly}(n))$. Thus the final time is $\exp(\text{poly}(n))\,\text{poly}\log(1/\epsilon)$.

KKT Ideal

Definition (KKT Ideal & Variety)

$$I_K = \left\{ v(x) f_1(x) + \sum h_{ij}(x) g_{ij}(x) \right\} =< f_1(x), g_{ij}(x) > .$$

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Main Theorems

Theorem (Zero-dimensional of generic I_K)

For a generic M, $|V(I_K)| < \infty$ and I_K is zero-dimensional.

Theorem (Degree bound)

There exists $m = O(\exp(\text{poly}(n)))$, s.t. for a generic M, $\epsilon > 0$,

$$v - f_0(x) + \epsilon = \sigma(x) + g(x),$$

where $\sigma(x)$ is SoS and $\deg(\sigma(x)) \leq m, g(x) \in I_K^m$.

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Estimate $h_{\text{ProdSym}(n)}(M)$ for a generic M to error ϵ needs $\exp(\text{poly}(n))\text{poly}\log(1/\epsilon)$.

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Arbitrary input M

Observations

- Generic M is *dense*. The opt of SDP could be continuous.
- Issue: SOS SDP might be *infeasible* up to degree m for arbitrary input M.

Solutions

- Switch to the dual SDP (moment): satisfies Slater's condition, i.e., strictly feasible.
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Proof of Theorem 1

Let $\mathcal{U} = \{f_1(x) = 0\}, \mathcal{W} = \{\forall\, i, j, g_{ij} = 0\}$. then $V(I_K) \subseteq \mathcal{U} \cap \mathcal{W}$.

It suffices to show $|\mathcal{U} \cap \mathcal{W}| < \infty$. Construct $\mathcal{A} = \mathcal{X} \cap \mathcal{U}$ s.t.

$\mathcal{A} \cap \mathcal{W} = \emptyset$ and $\dim(\mathcal{X}) = n - 1$. Note $\mathcal{W} \cap \mathcal{A} = (\mathcal{W} \cap \mathcal{U}) \cap \mathcal{X}$.

By Bézout's theorem, two varieties with dimension sum$\geq n$ must intersect. Thus

$$\dim(\mathcal{W} \cap \mathcal{U}) + \dim(\mathcal{X}) = \dim(\mathcal{W} \cap \mathcal{U}) + n - 1 < n.$$

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The Jacobian matrix $J_{\mathcal{A}} = \begin{pmatrix} \frac{\partial f_0}{\partial x_1} & \frac{\partial f_1}{\partial x_1} \\ \vdots & \vdots \\ \frac{\partial f_0}{\partial x_n} & \frac{\partial f_1}{\partial x_n} \end{pmatrix}$ has $\operatorname{rank}(J_{\mathcal{A}}) = 2$.

$\mathcal{W}$ by definition says $\operatorname{rank}(J_{\mathcal{A}}) = 1$. Thus no intersection!

Subtly: **genericity**; **projective space**; **homogenization**!

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Proof of Theorem 2

Let $\{\gamma_i\}$ be a Grobner basis for I_K.

$$|V(I_K)| < \infty \implies \max \deg\{\gamma_i\} \leq D = \exp(\mathrm{poly}(n)).$$

Now, want to bound $\deg(\sigma(x)), \deg(g(x))$ in

$$v - f_0(x) = \sigma(x) + g(x). \text{ s.t. } \sigma(x) \text{ SOS }, g(x) \in I_K^m.$$

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Perspectives

DPS+

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- KKT constraints from optimization theory.
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Open Questions

DPS+

- Analyze the low levels of DPS+.
- Advantages of adding KKT conditions other than presented here.

NPA+

- The use of NC KKT conditions.
- Can we have finite convergence for the field value?

SoS hierarchy

- Any other applications to quantum information?

Question And Answer

Thank you!

Q & A