Limitations of monogamy, Tsirelson-type bounds, and other SDPs in quantum information

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SDPs in Quantum Information

Semidefinite Programmings (SDPs) admit *polynomial time* solvers and plays an important role in quantum information.

- Consistency of reduced states, Quantum conditional min-entropy, local Hamiltonians
- QIP=PSPACE, QRG=EXP, ...

This talk is, however, about its **limitation** in

- Separability or entanglement detection,
- Approximation of Bell-violation (non-local game values).

Result: unconditional limitations of SDPs comparing to existing computational hardness.

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Problem 1: Separability

Definition (Separable and Entangled States)

A bi-partitie state $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ is *separable* if $\exists$ dist. $\{p_i\}$,

$$\rho = \sum p_i \sigma_X^i \otimes \sigma_Y^i, \text{ s.t. } \sigma_X^i \in \mathrm{D}(\mathcal{X}), \sigma_Y^i \in \mathrm{D}(\mathcal{Y}).$$

Otherwise, ρ is *entangled*. Let Sep $\stackrel{\text{def}}{=}$ { separable states }.

Definition (Entanglement Detection)

A KEY problem: given the description of $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$, decide

Either $\rho \in$ Sep, or ρ is far away from Sep.

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Alternative Formulation

Definition (Weak Membership)

WMem($\epsilon, \|\cdot\|$) : for any $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$, decide either $\rho \in$ Sep or $\|\rho - \text{Sep}\| \geq \epsilon$.

Via standard techniques in convex optimization, equivalent to

Definition (Weak Optimization)

WOpt(M, ϵ) : for any $M \in \mathrm{Herm}(\mathcal{X} \otimes \mathcal{Y})$, estimate the value of

$$h_{\mathrm{Sep}(d,d)}(M) := \max_{\rho \in \mathrm{Sep}} \langle M, \rho \rangle,$$

with additive error ϵ.

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$h_{\mathrm{Sep}(d,d)}(M)$

$$h_{\mathrm{Sep}(d,d)}(M) := \max_{\substack{x,y\in\mathbb{C}^d \\ \|x\|_2=\|y\|_2=1}} \sum_{i,j,k,l\in[d]} M_{ij,kl} x_i^* x_j y_k^* y_l. \tag{1}$$

REMARK: this is an instance of *polynomial optimization* problems with a homogenous degree 4 objective polynomial and a degree 2 constraint polynomial.

Connections

Quantum Information:

- *Mean-field* approximation in statistical quantum mechanics.
- *Positivity* test of quantum channels.
- Data hiding, Channel capacities, Privacy, ...
- *17 more examples* in quantum information in [HM10].

Quantum Complexity:

- Quantum Merlin-Arthur Game with Two-Provers (QMA(2)).

Classical Complexity:

- Unique Game Conjecture and Small-set Expansion. ($\ell_2 \to \ell_4$ norm)

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Heuristics

Separability Criterions:

- Positive Partial Transpose (PPT) : $\rho^{T_{\mathcal{Y}}} = \rho$? [PH]
- Reduction Criterions: $I_{\mathcal{X}} \otimes \rho_Y \geq \rho$? [HH]
- **FAILURE**: any such test has arbitrarily large error. [BS]

Doherty-Parrilo-Spedalieri (DPS) hierarchy:

- ρ is k-extendible if $\exists$ *symmetric* $\sigma \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_k)$, $\forall i, \rho = \sigma_{XY_i}$.
- $\rho \in \mathrm{Sep}$ if and only if ρ is k-extendible for any $k > 0$.
- Semidefinite program (SDP): size exponential in k.

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Computational Hardness

reference	k	c	s	n
GNN12	2	1	$1 - \frac{1}{d \cdot \mathrm{poly}\log(d)}$	$O(d)$
Per12	2	1	$1 - \frac{1}{\mathrm{poly}(d)}$	$O(d)$
AB+08	$\sqrt{d} \cdot \mathrm{poly}\log(d)$	1	0.99	$O(d)$
CD10	$\sqrt{d} \cdot \mathrm{poly}\log(d)$	$1 - 2^{-d}$	0.99	$O(d)$
HM13	2	1	0.01	$\frac{\log^2(d)}{\mathrm{poly}\log(d)}$

Table: Hardness results for $h_{\mathrm{Sep}^k(d)}$ (extension of $h_{\mathrm{Sep}(d,d)}$ to k parties.)

Hardness in the following sense: determining satisfiability of 3-SAT instances with n variables and $O(n)$ clauses can be reduced to distinguishing between $h_{\mathrm{Sep}^k(d)} \geq c$ and $\leq s$ as above.

Computational Hardness

Exponential Time Hypothesis (ETH)

The 3-SAT problem with n variables requires $2^{\Omega(n)}$ time to solve.

- Combine with [HM13] hardness result $\Rightarrow$ approximation of $h_{\mathrm{Sep}(d)}$ with constant precision requires $d^{\Omega(\log(d))}$ time.
- A matching upper bound: DPS to $O(log(d)/\epsilon^2)$ level for **1-LOCC** M: time $d^{O(\log(d)/\epsilon^2)} \to d^{O(\log(d))}$. [BYC, BH]

Question: any unconditional lower bounds for DPS or any SDPs? any matching upper bounds?

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Problem 2: Non-local Games

Non-local Game (denoted G):

- Two physically **separated** players Alice and Bob. **No** communication once the game starts.
- Sets of questions S, T and answers A, B and a distribution $\pi : S \times T \to [0, 1]$.
- Sample $(s, t) \in S \times T \sim \pi$ and ask Alice and Bob respectively. Obtain answers $a \in A, b \in B$.
- Determine **win** or **lose** by a predicate $V(a, b|s, t) \in \{0, 1\}$.

Motivation: Bell-violation (quantum **non-locality**) in a game language. Also related to **quantum multi-prover interactive proofs** with shared entanglement.

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Strategies:

- Denote by $P[a, b|s, t]$ the probability of answering (a, b) upon receiving (s, t).
- Quantum strategies: share a quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and answer w.r.t measurements $\{A_s^a\}$ and $\{B_t^b\}$,

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Definition (Game Value)

$$\omega(G) = \max_P \sum_{a,b,s,t} \pi(s,t) V(a,b|s,t) P(a,b|s,t).$$

Example: CHSH game:

- $A = B = S = T = \{0,1\}$ and $\pi(s,t) = 1/4, \forall (s,t) \in S \times T$.
- $V(a,b|s,t) = 1$ iff $a \oplus b = s \wedge t$.
- **Classical strategies:** $\omega(CHSH) = 3/4$. **Quantum strategies:** $\omega^*(CHSH) = \cos^2(\pi/8) \approx 0.85$.
- Quantum strategies are strictly more powerful.

Question: calculate $\omega^*(G)$ for any given G. How hard is that?

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$\omega^*(G)$ **for quantum strategies**: an optimization problem!

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- $\omega^*(G)$ is not known to be **computable**.
- A SDP hierarchy proposed by Navascues-Pironio-Acin (NPA) approximates $\omega^*(G)$ from above and converges at infinity.
- Converging rate only known for special cases: XOR, Unique games. No general upper or lower bounds known about the NPA hierarchy.

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IKM09	2	1	$1 - \frac{1}{\text{poly}(Q)}$	$O(Q)$
IV12	4	1	$2^{-Q^{\Omega(1)}}$	$Q^{\Omega(1)}$
Vid13	3	1	$2^{-Q^{\Omega(1)}}$	$Q^{\Omega(1)}$

Table: Hardness results for $\omega^*(G)$ where G is a one-round k-prover interactive proof protocol with question alphabet size Q.

Hardness in the following sense: determining satisfiability of 3-SAT instances with n variables and $O(n)$ clauses can be reduced to distinguishing between $\omega^*(G) \geq c$ and $\leq s$ as above.

Result I: Unconditional Hardness for h_{Sep}?

Will the hardness of $h_{\mathrm{Sep}(d)}$ for const ϵ hold w/o ETH?

Theorem (Main I.1)

The DPS hierarchy (or general Sum-of-Squares SDP) requires $\Omega(\log(d))$ levels to solve $h_{\mathrm{Sep}(d)}$ with constant precision.

Theorem (Main I.2)

Any SDP relaxation that estimate $h_{\mathrm{Sep}(d)}(M)$ with constant errors requires size $d^{\Omega(\log(d))}$.

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Result II: Unconditional Hardness for $\omega^*(G)$?

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Theorem (Main II.1)

There exists a family of games $\{G_n\}$ s.t. the NPA hierarchy requires $\Omega(n)$ levels to distinguish $\omega^*(G) = 1$ from $\omega^*(G) = 1 - \Omega(1/n^2)$.

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Any SDP relaxation that estimates $\omega^*(G)$ with precision $O(1/n^2)$ requires size $(n/\log(n))^{\Omega(n)}$.

Remark: Match the computational hardness of [IKM].
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- First unconditional lower bounds (SoS or SDP) for both h_{Sep} and $\omega^*(G)$ problems.
- Match all bounds from computational hardness for h_{Sep}, especially the DPS hierarchy.
- Improve lower bounds from level $k \leq 5$ to $k = \Omega(n)$ for the NPA hierarchy.

Consequences

- Implication to any resolution of the complexity of QMA(2). Discussed later.
- Hardness extends to the $2 \to 4$ norm, and thus small-set expansions (SSE), and potentially the unique game conjecture (UGC).

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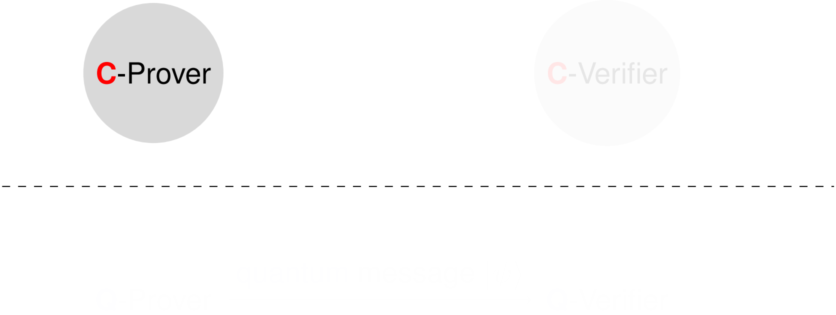
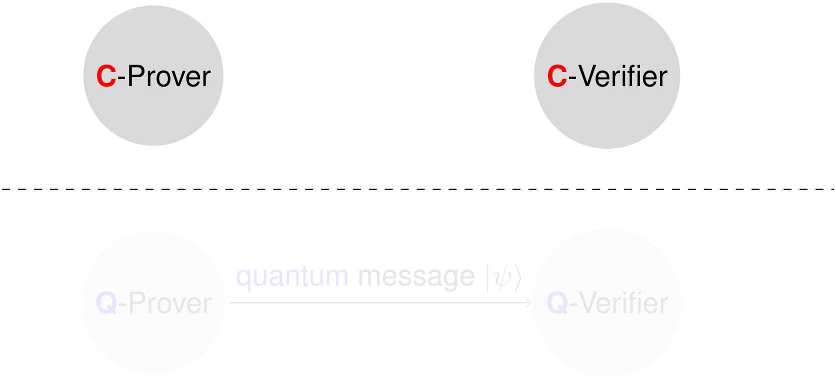
QMA(2) vs QMA



QMA(2) vs QMA


C-Prover
C-Verifier
Q-Prover
quantum message $|\psi\rangle$
Q-Verifier

QMA(2) vs QMA

C-Prover — classical message m_1 $\longrightarrow$ C-Verifier

NP

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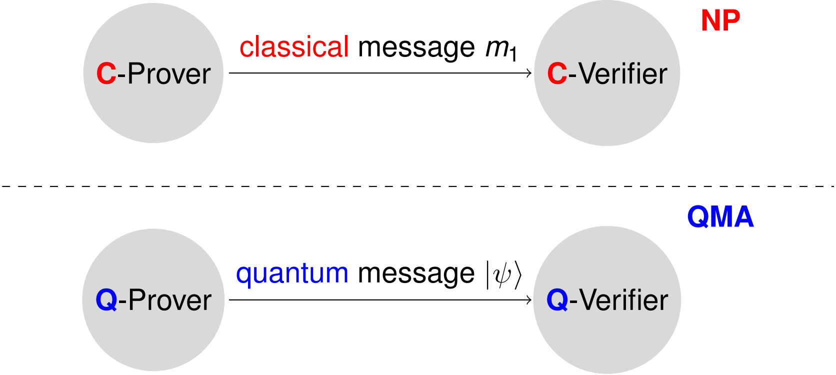
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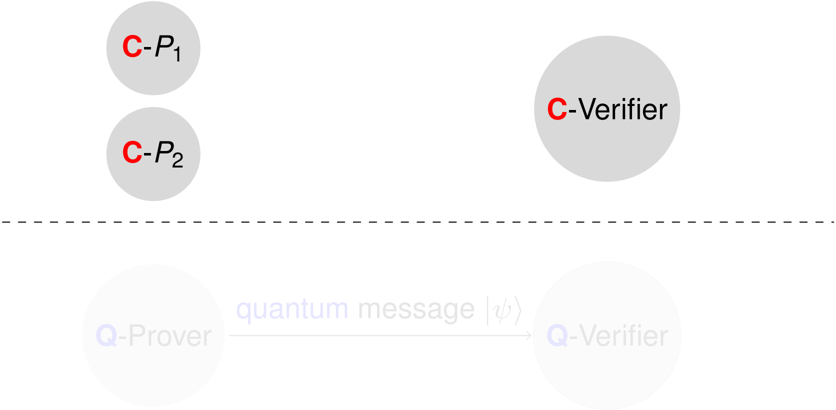
C-P_1
C-P_2
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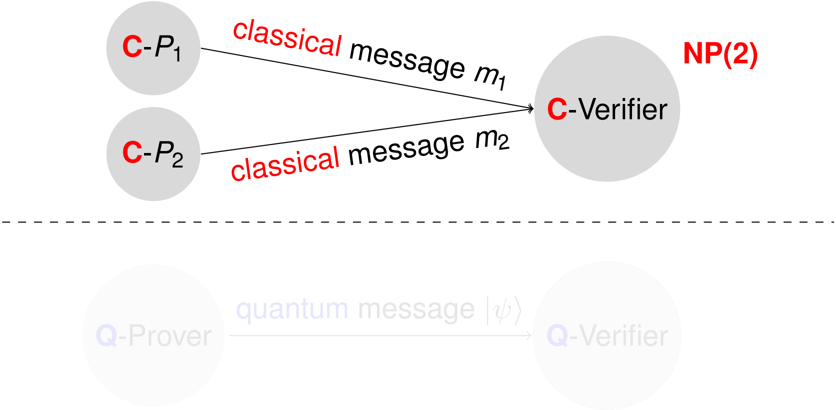
C-P_1
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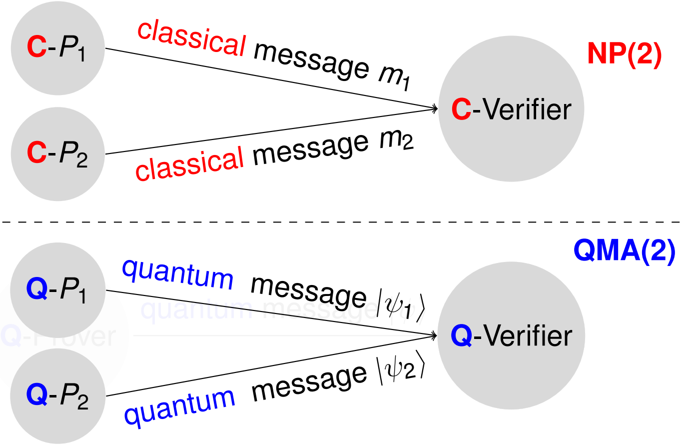
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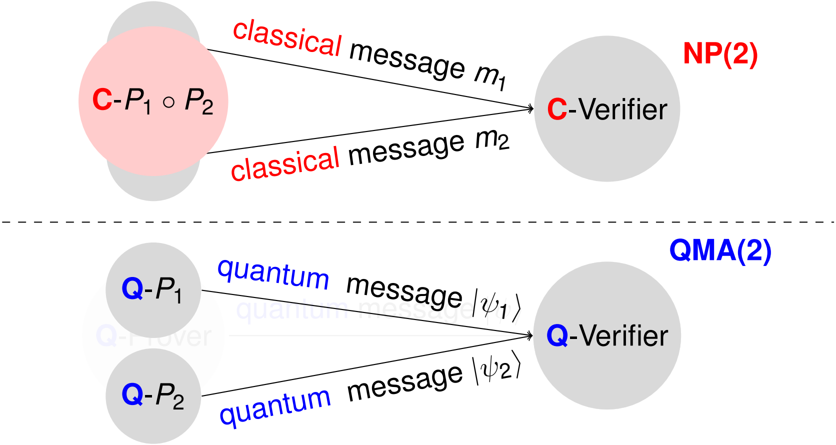
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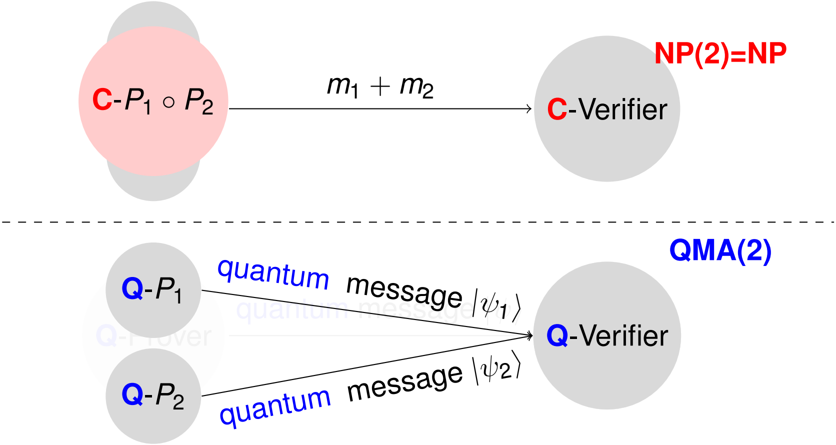
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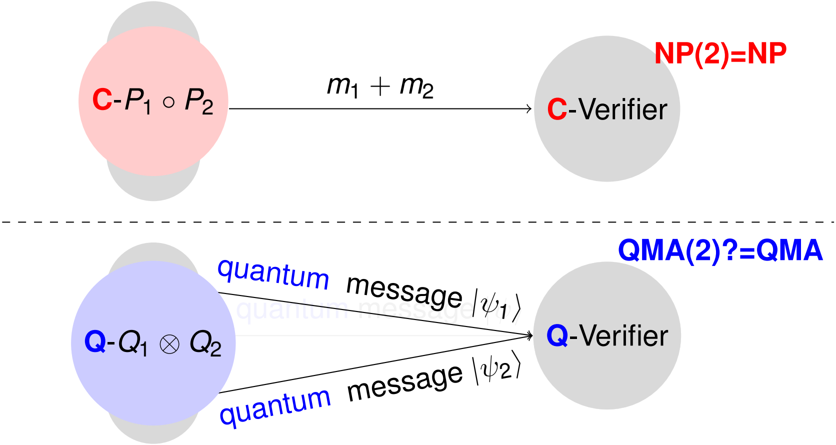
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- First study in [KMY01, KMY03]. Surprising: NP $\subseteq$ QMA$(2)_{\log}$ [BT09] v.s. QMA$_{\log}$ $=$ BQP [MW05].

Main open question is to improve the trivial upper bound NEXP.

- Variants of QMA(2) have been studied, such as BellQMA, LOCC-QMA, which collapse to QMA [Bra, ABD+, BCY].
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Hardness applies to QMA(2)

- Our explicit hard instance is a **valid** QMA(2) instance.
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Unconditional proof of Watrous's dis-entangler conjecture

- Dis-entangler: a hypothetical channel that a) its output is always ϵ-close to a separable state, and b) its image is δ-close to any separable state, both in trace distance.
- Input dimension $\dim(\mathcal{H}) = \infty$ for $\epsilon = \delta = 0$ [AB+09].
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Technical Target

- Introduce hardness of SDPs/SoS into quantum problems.
- Deal with their limitations, such as boolean domains and commutative problems.

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- Formulate a framework of reductions for this purpose. Also applicable to other problems, e.g., Nash's equilibria.
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Principle of Sum-of-Squares

One way to show that a polynomial $f(x)$ is *nonnegative* could be

$$f(x) = \sum a_i(x)^2 \geq 0.$$

Example

$$\begin{aligned} f(x) &= 2x^2 - 6x + 5 \\ &= (x^2 - 2x + 1) + (x^2 - 4x + 4) \\ &= (x-1)^2 + (x-2)^2 \geq 0. \end{aligned}$$

Such a decomposition is called a *sum of squares (SOS) certificate* for the non-negativity of f. The min degree, $\deg_{\mathrm{sos}}$.

Principle of SoS : constrained domain

Definition (Variety)

A set $V \subseteq \mathbb{C}^n$ is called an *algebraic variety* if $V = \{x \in \mathbb{C}^n : g_1(x) = \cdots = g_k(x) = 0\}$.

Non-negativity of $f(x)$ on V could be shown by

$$f(x) = \sum a_i(x)^2 + \sum b_j(x)g_j(x) \geq 0.$$

Question: whether all nonnegative polynomials on certain variety have a SOS certificate? Hilbert 17th problem!

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SoS in Optimization

$$\begin{aligned} \text{max} \quad & f(x) \\ \text{subject to} \quad & g_i(x) = 0 \quad \forall i \end{aligned} \tag{2}$$

is equivalent to (justified by *Positivstellensatz*)

$$\begin{aligned} \text{min} \quad & \nu \\ \text{such that} \quad & \nu - f(x) = \sigma(x) + \sum_i b_i(x) g_i(x), \end{aligned} \tag{3}$$

where $\sigma(x)$ is SOS and $b_i(x)$ is any polynomial.

SoS relaxation: Lasserre/Parrilo Hierarchy

- If $\sigma(x), b_i(x)$ have *any* degrees (or $\mathrm{deg}_{\mathrm{sos}}(v - f)$), then problem (3) is equivalent to problem (2).
- By bounding the degrees, we get the Lasserre/Parrilo hierarchy, which is a SDP hierarchy.

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Recall $h_{\mathrm{Sep}(d,d)}(M)$

$$h_{\mathrm{Sep}(d,d)}(M) := \max_{\substack{x,y\in\mathbb{C}^d \\ \|x\|_2=\|y\|_2=1}} \sum_{i,j,k,l\in[d]} M_{ij,kl} x_i^* x_j y_k^* y_l. \quad (5)$$

Recall: this is an instance of *polynomial optimization* problems with a homogenous degree 4 objective polynomial and a degree 2 constraint polynomial.

Its Lasserre's hierarchy is the DPS hierarchy with full symmetry.

Non-commutative (nc) SoS

Given $F, G_1, \ldots, G_m \in \mathcal{R}\langle X \rangle$, define

$$F_{\max} := \sup_{\rho, X=(X_1,\ldots,X_n)} \operatorname{Tr}[\rho F(X)]$$
$$\text{subject to } \rho \geq 0, \operatorname{Tr}\rho = 1, G_1(X) = \cdots = G_m(X) = 0. \tag{6}$$

Note that the supremum here is over density operators ρ and Hermitian operators $X_1, \ldots, X_n$ that may be infinite dimensional;

ncSoS

A non-commutative SoS proof can be expressed similarly as

$$c - F = \sum_{i=1}^{k} P_i^\dagger P_i + \sum_{i=1}^{m} Q_i G_i R_i, \tag{7}$$

for $\{P_i\}, \{Q_i\}, \{R_i\} \subset \mathcal{R}\langle X \rangle$. Likewise the best degree-d ncSoS proof can be found in time $n^{O(d)} m^{O(1)}$ by SDPs.

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General SDPs

- The DPS and NPA hierarchies are just SoS and ncSoS SDP hierarchies.
- Thus, lower bounds for $\deg_{\text{sos}}$ $\Rightarrow$ lower bounds for DPS and NPA.
- How about general SDPs?

Lee-Raghavendra-Steurer

- Any $\deg_{\text{sos}}$ lower bound on $\{0,1\}^n$ $\Rightarrow$ a lower bound on SDP relaxations.
- SDP relaxation: $\forall x \in \{0,1\}^n$, $\exists$ relaxed X^x s.t. $f(x) = F(X^x)$. Embedding!
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Pseudo-distribution

Dual of the SOS cone

- Let $\Sigma_{d,2D}$ be the cone of all PSD matrices representing SOS polynomials with degree up to $2D$.
- The dual cone $\Sigma^*_{d,2D}$ is moment $M_D(x) \geq 0$, where entry (α, β) of $M_d(x)$ is $\int x^{\alpha+\beta}\mu(dx), |\alpha|, |\beta| \leq d$.

Pseudo-distributrion/expectation

- Moment $M_D(x)$ gives rise to *pseudo-distribution*. Expectation on it is *pseudo-expectation*.
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Pseudo-expectation

A degree-d pseudo-expectation $\tilde{\mathbb{E}}$ is an element of $\mathcal{R}[x]_d^*$ (i.e. a linear map from $\mathcal{R}[x]_d$ to $\mathcal{R}$) satisfying

- **Normalization.** $\tilde{\mathbb{E}}[1] = 1$.
- **Positivity.** $\tilde{\mathbb{E}}[p^2] \geq 0$ for any $p \in \mathcal{R}[x]_{d/2}$.

$\tilde{\mathbb{E}}$ satisfies the constraints $g_1, \ldots, g_m$ if $\tilde{\mathbb{E}}[g_i q] = 0$ for all $i \in [n]$ and all $q \in \mathcal{R}[x]_{d-\deg(g_i)}$.

$$f_{\mathrm{SoS}}^d = \max\{\tilde{\mathbb{E}}[f] : \tilde{\mathbb{E}} \text{ of degree-}d \text{ satisfying } g_1, \ldots, g_m\}. \quad (8)$$

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Integrality Gaps

What constitutes an integrality gap?

- An instance Φ that has $f_{\text{opt}}(\Phi)$ is small.
- But $f^d_{\text{SoS}}(\Phi)$ is large for some $d \Rightarrow$ lower bound at level d.

Example

- 3XOR: $O(n)$ clauses on n boolean variables: $x_i \oplus x_j \oplus x_k = C_{ijk}$.
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Extend integrality gaps via reductions

Reduction from A to B

- Reduction is an instance-mapping $\Phi^A \to \Phi^B$.
- **Soundness**: $f_{\text{opt}}^A(\Phi^A)$ small $\Rightarrow f_{\text{opt}}^B(\Phi^B)$
- **Pseudo-completeness**: $f_{\text{SoS}}^{d_A}(\Phi^A)$ large $\Rightarrow f_{\text{SoS}}^{d_B}(\Phi^B)$ large, d_B is not too smaller than d_A.

Pseudo-completeness: low-degree reduction

- Let $\mu_A(\bar{\mathbb{E}}_A)$ be the pseudo-solution for Φ^A. One needs to construct a $\mu_B(\bar{\mathbb{E}}_B)$ for Φ^B.
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A reduction with *pseudo-completeness* and *soundness* leads to an integrality gap of degree d_B for Φ^B.

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- Only apply to $\{0,1\}^n \Rightarrow$ no direct application on h_{Sep} or $\omega^*(G)$.
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A typical reduction

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- Reductions $R_1, \cdots, R_2$ lead to an SoS integrality gap at the problem A.
- Apply LRS on the problem A over boolean domains.
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Real reductions for h_{Sep} and $\omega^*(G)$

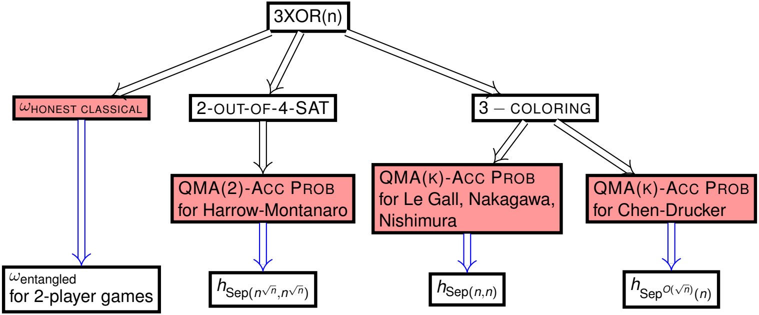

Figure: All our results are derived from the integrality gaps of 3XOR. **Red nodes**: problems over the boolean cube and LRS is applied. **Blue arrows** are "embedding reductions".

Reduction for h_{Sep}

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- R_1: a classical step. Low-degree & soundness similar to the degree reduction step in Dinur's proof of the PCP theorem.
- R_2: a quantum step. Apply a modified QMA(2) protocol for 3-SAT [AB+09, HM13]. Low-degree due to the tests of the protocol. Soundness inhered from the protocol.
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- Match ETH-based bounds for h_{Sep}.
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Question And Answer

Thank you!

Q & A

SoS relaxation: Lasserre/Parrilo Hierarchy

- If $\sigma(x), b_i(x)$ have *any* degrees (or $\mathrm{deg}_{\mathrm{sos}}(v - f)$), then problem (3) is equivalent to problem (2).
- By bounding the degrees, we get the Lasserre/Parrilo hierarchy.

$$\begin{aligned} &\min && \nu \\ &\text{such that} && \nu - f(x) = \sigma(x) + \sum_i b_i(x) g_i(x), \end{aligned} \tag{9}$$

where $\sigma(x)$ is SOS and $b_i(x)$ is any polynomial and $\deg(\sigma(x))$, $\deg(b_i(x)g_i(x)) \leq 2D$.

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Why it is a SDP?

Observation

- Any $p(x)$ (of degree $2D$) $= m^T Qm$, where m is the vector of monomials of degree up to $2D$ and Q is the coefficients.
- $p(x)$ is a SOS iff $Q \geq 0$.

$$\begin{aligned} \min_{\nu, b_{i\alpha} \in \mathbb{R}} \quad & \nu \\ \text{such that} \quad & \nu A_0 - F - \sum_{i\alpha} b_{i\alpha} G_{i\alpha} \geq 0. \end{aligned} \tag{10}$$

Complexity: $\mathrm{poly}(m)\,\mathrm{poly}\log(1/\epsilon)$, where $m = \binom{n+D}{D}$.

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- Any $p(x)$ (of degree $2D$) $= m^T Qm$, where m is the vector of monomials of degree up to $2D$ and Q is the coefficients.
- $p(x)$ is a SOS iff $Q \geq 0$.

$$\begin{aligned} \min_{\nu, b_{i\alpha} \in \mathbb{R}} \quad & \nu \\ \text{such that} \quad & \nu A_0 - F - \sum_{i\alpha} b_{i\alpha} G_{i\alpha} \geq 0. \end{aligned} \tag{10}$$

Complexity: $\mathrm{poly}(m)\,\mathrm{poly}\log(1/\epsilon)$, where $m = \binom{n+D}{D}$.

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