CMSC452 Elementary Theory of Computation, Fall 2001

Sketch of Solutions: Homework 5

Note: 5 partial credits are given to each graded unit for minor errors.

Problem 3.4.1 [10+10=20 points]
The corresponding machine is \( M = \{ \{ p, q \}, \{ \lambda \}, \{ \lambda \}, \Delta, \{ \lambda \} \} \), where \( \Delta = \{(p, \alpha, \beta), (q, \gamma)\} \) and the transition relation \( \lambda : \Gamma \to \Gamma \). Then \( \Delta \) is the set of all possible transitions.

Then the acceptance of the input string \( \lambda((p, \alpha, \beta), (q, \gamma)) \) is determined by the following rules:

1. Given transition relation \( \lambda((p, \alpha, \beta), (q, \gamma)) \), if both \( \beta \neq e \) and \( \gamma \neq e \), replace it by \( ((p, \alpha, \beta), (q, \gamma)) \), where \( r \) is a new state. [10 points]

2. For each transition relation \( \lambda((p, \alpha, \beta), (q, \gamma)) \), if \( |\gamma| > 1 \), it can be replaced by \( ((p, \alpha, \beta), (q, \gamma_1)) \) and \( ((r, e), (q, \gamma_2)) \), where \( \gamma_1 = \gamma \gamma_2 \) with \( |\gamma_1| \geq 1 \) and \( |\gamma_2| \geq 1 \) and \( r \) is a new state. After \( |\beta| \) such operations, the transition relation can be replaced by \( |\beta| \) others with unit \( \beta \)’s. Similarly, Each transition relation with \( |\gamma| > 1 \) can be replaced by some others with unit \( \gamma \)’s. [10 points]

Problem 3.5.1 (a) [10 points]
Let \( L_1 = \{ a \} \cdot \{ a^m b^n : n \in \mathbb{N} \} = \{ a^m b^n : m > n \} \). Then \( L_1 \) is context-free since it is the concatenation of two context-free languages. Similarly, \( L_2 = \{ a^n b^m : n \in \mathbb{N} \} \{ b \} + = \{ a^m b^n : m > n \} \) is also context-free. Finally, \( \{ a^m b^n : m \neq n \} \cup \{ a^m b^n : m > n \} = L_1 \cup L_2 \) is also context-free.

Problem 3.5.2 (c) [10 points]
Suppose \( L = \{ w w w : w \in \Sigma^* \} \) is context-free. By theorem 3.5.3, there is \( k > 0 \) such that for any \( w \in L \) with \( |w| \geq k \), \( w \) can be represented by \( u v x y z \) where \( u, v, x, y, z \in \Sigma^* \), \( |v x y| \leq k \) \( v y z \) such that \( u^k v^k y^k z \in L \) for \( n \geq 0 \).

1. Consider the string \( w = a^k b^k a^k b^k \) with \( |w| \geq k \). It can be represented by \( w = u v x y z \) as above.

2. Neither \( v \) nor \( y \) can contain \( b \). Otherwise, \( u^k v^k y^k z \notin L \) for \( n \geq 4 \).

3. Thus, \( v, y \in \{ a^*: |a^*| \leq k \} \). Let \( v = a^p \) and \( y = a^q \). for \( p, q \leq k \). Suppose \( v \) is in the first \( a \)'s and \( y \) is the second \( a \)’s. Then \( u^k v^k y^k z = a^k b^k a^{k+s} b^k a^k b^k \notin L \). In general, other cases will result in \( u^k v^k y^k z = a^{k+r} b^k a^{k+s} b^k a^{k+t} b^k b^k b^k b^k \) such that \( r, s, t \in \{ 0, p, q, p + q \} \) and \( r + s + t = p + q \). All the possibilities cannot be in \( L \). Thus \( L \) is not context-free by contradiction.
Problem 3.5.14 (b)(d) [10+10=20 points]

(b) \( \{a^nb^n c^n : m\neq n \text{ or } n\neq p \text{ or } p\neq m\} \) is context-free. \( \{a^nb^n c^n : m\neq p\} \) is context-free because it is essentially of the form of problem 3.5.1. Similarly, \( \{a^nb^n c^n : n\neq p\} \) and \( \{a^nb^n c^n : p\neq m\} \) are context-free. Thus, \( \{a^nb^n c^n : m\neq n \text{ or } n\neq p \text{ or } p\neq m\} = \{a^nb^n c^n : m\neq n\} \cup \{a^nb^n c^n : n\neq p\} \cup \{a^nb^n c^n : p\neq m\} \), a union of context-free languages, is also context-free. [10 points]

(d) \( L = \{w \in \{a,b,c\}^* : w \) does not contain equal numbers of occurrences of \( a, b, \) and \( c\} \) is context-free. Let \( L_1 = \{w \in \{a,b,c\}^* : \) the number of occurrence of \( a \) is different from that of \( b\} \), \( L_2 = \{w \in \{a,b,c\}^* : \) the number of occurrence of \( b \) is different from that of \( c\} \), and \( L_3 = \{w \in \{a,b,c\}^* : \) the number of occurrence of \( c \) is different from that of \( a\} \). Since \( L = L_1 \cup L_2 \cup L_3 \), a union of context-free languages, \( L \) is context-free.

Problem 3.5.15 [10+10=20 points]

\( L - R = \{w \in \{a,b,c\}^* : w \) has an equal number of \( a, b, \) and \( c\} \) is context-free since it is the intersection of context-free language \( L \) and regular language \( R = \sum^* - R \). [10 points]

\( R - L \) is not necessarily context-free. For example, let \( R = \sum^* \) which is regular. Then \( R - L = L' \) needs not to be context-free since context-free languages are not closed under complementation. [10 points]