CMSC 250 Exam 1 answers, Fall 2001

1. (20pts)

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>\sim p</th>
<th>\sim q</th>
<th>\sim r</th>
<th>q \lor r</th>
<th>\sim (1)</th>
<th>\sim p \lor (2)</th>
<th>\sim q \land \sim r</th>
<th>p \rightarrow (4)</th>
</tr>
</thead>
<tbody>
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The two statements are logically equivalent because columns (3) and (5) are the same.

2. (10pts)

(a) 00001001
(b) 1111011
(c) 00000100
(d) 4

3. (20pts)

<table>
<thead>
<tr>
<th>line</th>
<th>Statement</th>
<th>Reason</th>
<th>Lines</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>p \rightarrow r</td>
<td>Defn. of conditional</td>
<td>P3</td>
</tr>
<tr>
<td>2</td>
<td>\sim \sim r \lor \sim q</td>
<td>De Morgan’s</td>
<td>P2</td>
</tr>
<tr>
<td>3</td>
<td>r \lor \sim q</td>
<td>Double Negation</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>\sim q \lor r</td>
<td>Commutative</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>q \rightarrow r</td>
<td>Defn. of Conditional</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>r</td>
<td>Division into cases</td>
<td>P1, 1, 5</td>
</tr>
</tbody>
</table>

There are many different ways to prove this. The above is one example, but the students might also assume p and \sim p, or assume q and \sim q, or assume p and q and show that all of these imply r. The students might also convert P2 and P3 to disjunctions, and then use the distributive law and a disjunctive syllogism.
4. (20 pts)

(a) Statement: \( \exists d \in \{ \text{dogs} \} \exists h \in \{ \text{houses} \} L(d, h) \).
    Negation: \( \forall d \in \{ \text{dogs} \} \forall h \in \{ \text{houses} \} \sim L(d, h) \).

(b) Statement: \( \forall x \in \mathbb{Z} \exists y \in \mathbb{Z} L(y, x) \).
    Negation: \( \exists x \in \mathbb{Z} \forall y \in \mathbb{Z} \sim L(y, x) \).

(c) Statement: \( \exists r \in \mathbb{R} \exists a, b \in \mathbb{Z} Q(r, a, b) \).
    Negation: \( \forall r \in \mathbb{R} \forall a, b \in \mathbb{Z} \sim Q(r, a, b) \).

(d) Statement: \( \forall c_1, c_2 \in \{ \text{cats} \} \forall h_1, h_2 \in \{ \text{houses} \} (L(c_1, h_1) \land L(c_2, h_2)) \rightarrow c_1 = c_2 \).
    Negation: \( \exists c_1, c_2 \in \{ \text{cats} \} \exists h_1, h_2 \in \{ \text{houses} \} (L(c_1, h_1) \land L(c_2, h_2) \land c_1 \neq c_2) \).

5. (10 pts)

(a) Invalid:

(b) Valid:

6. (20 pts)
\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
line & Statement & Reason & Lines \\
\hline
1 & \(M(a) \land Q(a)\) & \(\exists \text{ Inst.}\) & P1 \\
2 & \(M(a)\) & Conj. Simpl. & 1 \\
3 & \(R(a)\) & \(\forall \text{ MP}\) & 2, P2 \\
4 & \(\sim (\sim P(a) \land Q(a))\) & \(\forall \text{ Inst.}\) & P3 \\
5 & \(\sim P(a) \lor \sim Q(a)\) & De Morgan’s & 4 \\
6 & \(P(a) \lor \sim Q(a)\) & Double Neg. & 5 \\
7 & \(Q(a)\) & Conj. Simpl. & 1 \\
8 & \(P(a)\) & Disj. Syll. & 6, 7 \\
9 & \(R(a) \land P(a)\) & Conj. Add. & 3, 8 \\
10 & \(\exists w \in DR(w) \land P(w)\) & \(\exists \text{ Gen.}\) & 9 \\
\hline
\end{tabular}
\end{center}

7. There are many possible answers for this one.
Some include:
\[P(x, y, z) = [x] + [y] + [z] \neq x + y + z.\]

8. (20pts)

\text{Proof:}

Suppose that \(\sqrt[3]{7}\) is rational.
Then, \(\sqrt[3]{7} = \frac{a}{b}\) for some \(a, b \in \mathbb{Z}\) where \(b \neq 0\).
Hence, \(7 = \frac{a^3}{b^3}\), which means that \(7b^3 = a^3\).

By the unique factorization theorem,
\[a = 7^{k_a}a_1, \text{ where } k_a \in \mathbb{Z}^{\text{nonneg}}, \text{ and } 7 \nmid a_1.\]
\[b = 7^{k_b}b_1, \text{ where } k_b \in \mathbb{Z}^{\text{nonneg}}, \text{ and } 7 \nmid b_1.\]

Thus,
\[7b^3 = 7(7^{3k_b})b_1^3 \text{ and } a^3 = 7^{3k_a}a_1^3.\]
And, since \(7b^3 = a^3\),
\[7(7^{3k_b})b_1^3 = 7^{3k_a}a_1^3.\]

Because both sides are the same integer, and there is a unique factorization of that integer into primes, the powers of 7 must be the same on both sides.
Hence, \(7^{3k_b+1} = 7^{3k_a}\), which means that
$3k_b + 1 = 3k_a.$

Therefore, $3k_b - 3k_a = -1$, and $k_b - k_a = -\frac{1}{3}$.

But, this is a contradiction because $k_a$ and $k_b$ are integers, so their difference is also an integer.

Therefore, $\sqrt[3]{7}$ is not rational.

9. (20pts)

Proof:

Because $a \equiv b \mod n$, $a - b = nq_1$ for some $q_1 \in \mathbb{Z}$.

Because $b \equiv c \mod m$, $b - c = mq_2$ for some $q_2 \in \mathbb{Z}$.

Because $m|n$, $n = mk$ for some $k \in \mathbb{Z}$.

\[ a - c = a - b + b - c = (a - b) + (b - c). \]

\[ a - c = nq_1 + mq_2 = (mk)q_1 + mq_2 = m(kq_1 + q_2). \]

Hence, $a - c = mj$ for some $j \in \mathbb{Z}$, which means that $m|a - c$. 