Spring 2001

CMSC 250: Midterm 2

On this exam you are allowed one 8 1/2 x 11 inch piece of paper. You are NOT allowed: calculators, magnifying glasses, slide rules, PDAs or computers, textbooks, more than one page of notes, or telephones. TURN OFF ALL PHONES AND BEEPERS BEFORE THE EXAM BEGINS.

0. (5 points) Write your name, student ID number, AND section number on the front cover NEATLY.

1. (10 points)

(a) Find the first 6 terms of the following sequence:

\[ a_0 = 3 \]
\[ a_1 = 4 \]
\[ a_k = 2a_{k-1} + a_{k-2} \]

(b) Find the value of the following summation:

\[ \sum_{i=0}^{4} [2^i + i] \]

2. (15 points) Use mathematical induction to show that for \( n \geq 1 \),

\[ \sum_{i=1}^{n} (-1)^{i-1} i^2 = (-1)^{n-1} \frac{n(n+1)}{2} \]

3. (15 points) Let

\[ a_0 = 1 \]
\[ a_k = 2a_{\lfloor \frac{k}{2} \rfloor} + 3a_{\lfloor \sqrt{\frac{k}{2}} \rfloor} + 1 \]

where \( k > 0 \). Prove that \( \forall k \geq 0, a_k \equiv 1 \pmod{5} \).

4. (20 points) Prove that \( \sqrt[3]{\frac{1}{3}} \) is not rational. Prove all lemmas.

5. (15 points) Prove that there are an infinite number of positive composite integers \( x \) such that \( x \equiv 1 \pmod{4} \). Prove all lemmas.

6. (20 points) For each of the following, state whether the statement is true or false, and then prove or disprove it as appropriate.

(a) If \( a \equiv 3 \pmod{8} \) and \( b \equiv 7 \pmod{12} \), then \( ab + 9^a \equiv 2 \pmod{4} \).

(b) If \( a \equiv b \pmod{m} \) and \( x \equiv y \pmod{m} \), then \( a^x \equiv b^y \pmod{m} \).