CMSC 250          Exam #2          Thursday, April 25, 2002

Write all answers legibly on the paper provided. If you need extra paper, raise your hand and request a blank paper – you must put your name on and hand-in any paper you receive. The number of points possible for each question is indicated in square brackets – the total number of points on the exam is 150, and you will have exactly 1.5 hours to complete this exam. You may not use calculators, textbooks or any other aids during this exam.

1. [15 pts.] Assume there are 21 students in a class: 8 are boys and 13 are girls.
   Answer the following questions about those students. You do not have to do the arithmetic, but you do have to get each answer into a form that includes factorials and/or exponents: (Be sure to clearly label each of your answers.)

   a. How many different lines of the 21 students can be made (meaning order of students in a single row)?

   b. How many different 5 member teams can be made?

   c. How many different 5 member teams can be made if all the members of the team must be the same gender?

   d. How many different 5 member teams can be made if there must be 2 or 3 girls on the team?

   e. How many different 5 member teams can be made if there cannot be any boys on the team?

**** This area is for grading purposes (points lost per page) - Do not write below this line ****

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2. [12 pts.] In a certain school with 100 students, each student is enrolled in at least one of English (E), History (H), or Mathematics (M). Given the following data:

- \( n(E) = n(H) = n(M) = 50 \)
- \( n(E \cap H) = n(E \cap M) = n(H \cap M) = 2 \cdot n(E \cap M \cap H) \)

Answer each of the following questions:

a. How many students are taking all of the courses (English, History and Mathematics)?

b. How many students are taking Mathematics, but are not taking English nor History?

Put the single number answer in the space provided, but for any partial credit, you must show your work below this box.

To receive Partial Credit for an incorrect answer above, you must CLEARLY label your work.
3. [15 pts.] Either give a counter example to disprove (giving specific members for the sets A, B and C) or use only the rules provided on the “cheat sheet” along with any definitions from the textbook and/or class to prove the following statement.

Be sure to give the name of the reason which justifies each step you give.

\[(A \cup (B \cap C^c)) - A = (B - A) - C\]
4. [18 pts.] Let \( f, g : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^3 - 4, \ g(x) = \sqrt[3]{x+4} \) for all \( x \in \mathbb{R} \). (Remember: \( (f \circ g)(x) = f(g(x)) \))

a. Write the formula for \( f \circ g \).

b. Write the formula for \( g \circ f \).

c. Write the formula for \( (g \circ f)^{-1} \).

d. Write the formula for \( g^{-1} \).

e. Write the formula for \( f^{-1} \).

f. Write the formula for \( g^{-1} \circ f^{-1} \).
5. [20 pts.] Use induction on \( n \) to prove that the following is true. (Hint: You may use the fact proven in class and in the book that the sum of the first \( p \) integers, \( 1 + 2 + 3 + \cdots + p \), equals \( \frac{p(p+1)}{2} \).)

\[
\forall n \in \mathbb{Z}^\geq 1, \prod_{k=1}^{n} \left( \frac{1}{k} \left( \sum_{j=1}^{k} j \right) \right) = \frac{(n + 1)!}{2^n}
\]
6. [20 pts.] Let $S$ be a set with 10 distinct integers in it such the difference between the smallest and the largest integer in $S$ is 20. Prove that there must be at least two distinct pairs of elements of $S$ that have the same sum. Be sure to CLEARLY SHOW your work.
7. [25 pts.] Given the following facts:

\[ a_1 = 2, \ a_2 = 5, \ a_3 = 9, \ \text{and} \ \forall n \in \mathbb{Z}^{>3}, \ a_n = 4a_{n-1} + 3a_{n-2} + a_{n-3} \]

Prove that \( \forall n \in \mathbb{Z}^+, \ a_n \geq 2(n - 1)^2. \)
8. [25 pts.] Let $n$ be a positive integer. Given the following sets:

- Let $S$ be defined by:
  - Let $\Sigma$ be any alphabet with 4 elements.
  - Let $S$ be the Cartesian Product: $S = \Sigma^n \times \Sigma^n$.

- Let $T$ be defined by:
  - Let $A$ be any set with $2n$ elements.
  - Let $P(A)$ be the power set of $A$.
  - Let $T = \{ \{ x, y \} | x \in P(A) \land y \in P(A) \mathrm{~and~} x \neq y \}$ (which means $T$ is the set of unordered pairs of distinct elements from $P(A)$)

Prove that any function $f : S \rightarrow T$ must map at least three elements of $S$ to a single element of $T$. (Be sure to clearly label the sets you are working with and the sizes of each of those sets.)