1. Use mathematical induction to prove that $(\forall n \geq 1) \left[ \sum_{i=1}^{n} (i2^i) = (n - 1)2^{n+1} + 2 \right]

2. Use mathematical induction to prove that for $n \geq 0$, $[6 \mid (7^n - 1)]$.

3. Use mathematical induction to prove that $(\forall n \geq 1) \left[ \sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n} \right]

4. Define a sequence by:
   \[
   a_0 = 1 \\
   a_n = 3a_{\left\lfloor \frac{n}{4} \right\rfloor} + 5a_{\left\lceil \frac{n}{4} \right\rceil} + 9^n
   \]
   Prove that $\forall n \geq 0, [a_n \equiv 1 \pmod{4}]$.

5. Consider the following game of NIM: There is a pile of $n$ stones. You are playing with another player. On your turn you may remove either 1 or 4 stones (exactly), if possible. The other player also may remove either 1 or 4 stones (exactly). Each player alternates turns. You start first, then the other player, then you again, etc.
   (a) Assume that the player to take the last stone WINS. Develop a winning strategy for the game.
   (b) Use mathematical induction, to show that this strategy wins.
   (c) Now assume that the player to take the last stone LOSES. Develop a winning strategy for this new version.
   (d) Use mathematical induction, to show that this strategy wins for the new version.

6. A property $P$ of the rationals is said to be Bellian if and only if the following is true:
   - $(\forall x) \left[ P(x) \rightarrow P(x^3) \right]$.
   (yes, we have made up the term “Bellian”). A set of rationals is said to be Bellian if and only if it is the truth set of some Belian predicate.
   (a) If $P$ is Bellian, and $P(2)$ is true, then describe the entire set of values of $x$ for which $P$ must also be true. (Use truth set notation, i.e. $\{ x \in \mathbb{Z} \mid P(x) \}$, where $P(x)$ is an appropriate predicate).
   (b) Let $R(x)$ be “$x^2 = 1$”. Is $R$ Belian? Explain. lain.
   (c) Give two examples of infinite sets that are Bellian. NO EXPLANATION NEEDED.
   (d) Give two examples of infinite sets that are not Bellian. NO EXPLANATION NEEDED.

7. Remember the distributive law from chapter 1, for any logical statements $p$, $q_1$ and $q_2$.
   \[ p \land (q_1 \lor q_2) \iff (p \land q_1) \lor (p \land q_2) \]
   Use this law and mathematical induction to prove the (EVEN MORE) generalized distributive law: For all integers $n \geq 2$ if $p, q_1, \ldots, q_n$ are logical statements, and $Q$ is an arbitrarily and fully parenthesized disjunction of the $q_i$'s (for example $((q_1 \lor (q_2 \lor q_3)) \lor (q_4 \lor q_5))$) then $p \land Q$ is the disjunction of the conjunctions with $p$, with the same parentheses (For example $(p \land ((q_1 \lor (q_2 \lor q_3)) \lor (q_4 \lor q_5))) \iff (((p \land q_1) \lor ((p \land q_2) \lor (p \land q_3))) \lor ((p \land q_4) \lor (p \land q_5)))$. 