CMSC 250—Fall 2000
Solution to Midterm 2

1. 

\[ \{ x \mid x = 1 \lor \exists k \in \mathbb{Z}^+ \frac{1}{3^k} \} \]

This is the set \{1, 1/3, 1/9, \ldots, 1/3^k, \ldots\}.

2. Let \( P(n) \) be:

\[
P(n) : \sum_{i=1}^{n} \frac{1}{(2i - 1)(2i + 1)} = \frac{n}{2n + 1}
\]

**Base Case:** Prove \( P(1) \).

\[
\sum_{i=1}^{1} \frac{1}{(2i - 1)(2i + 1)} = \frac{1}{2 \cdot 1 + 1}
\]

\[
\frac{1}{(2 \cdot 1 - 1)(2 \cdot 1 + 1)} = \frac{1}{3} \quad \text{Expand} \sum
\]

\[
\frac{1}{(1)(3)} = \frac{1}{3} \quad \text{Simplify}
\]

**IH:** Assume \( P(k) \) for \( k \geq 1 \). That is, assume:

\[
\sum_{i=1}^{k} \frac{1}{(2i - 1)(2i + 1)} = \frac{k}{2k + 1}
\]

**Ind. Case:** Prove \( P(k + 1) \) That is, prove:

\[
\sum_{i=1}^{k+1} \frac{1}{(2i - 1)(2i + 1)} = \frac{k + 1}{2(k + 1) + 1}
\]
3. This was a clever induction. It's basically a summation of 1 to n done in the exponent. Main errors from the midterm—not noticing it was a product.

Let \( P(n) \) be:

\[
P(n) : \prod_{i=1}^{n} (9^i) = 3^{n^2+n}
\]

**Base Case:** Prove \( P(1) \).

\[
\prod_{i=1}^{1} (9^i) = 3^{1^2+1}
\]

\[
9^1 = 3^2 \quad \sqrt{\text{Expand } \prod \text{ and simplify}}
\]

**IH:** Assume \( P(k) \) for \( k \geq 1 \). That is, assume:

\[
\prod_{i=1}^{k} (9^i) = 3^{k^2+k}
\]
**Ind. Case:** Prove $P(k + 1)$ That is, prove:

\[
\prod_{i=1}^{k+1} (g^i) = 3^{(k+1)^2+(k+1)}
\]

\[
\prod_{i=1}^{k+1} (g^i) = 3^{(k+1)^2+(k+1)}
\]

\[
\prod_{i=1}^{k} (g^i) \times \prod_{i=k+1}^{k+1} (g^i) = 3^{(k+1)^2+(k+1)} \quad \text{(Split } \prod\text{)}
\]

\[
\prod_{i=k+1}^{k+1} (g^i) = \frac{3^{(k+1)^2+(k+1)}}{\prod_{i=1}^{k} (g^i)} \quad \text{(Move } \prod \text{ to RHS)}
\]

\[
g^{k+1} = \frac{3^{(k+1)^2+(k+1)}}{\prod_{i=1}^{k} (g^i)} \quad \text{(Simplify } \prod \text{ on LHS)}
\]

\[
g^{k+1} = \frac{3^{(k+1)^2+(k+1)}}{3^{k^2+k}} \quad \text{(Use IH on } \prod\text{)}
\]

\[
(3^2)^{k+1} = \frac{3^{(k^2+2k+1)+(k+1)}}{3^{k^2+k}} \quad \text{(Rewrite 9 as } 3^2 \text{ and expand exp.)}
\]

\[
3^{2(k+1)} = 3^{2k+2} \quad \text{(Divide on RHS)}
\]

4. Let $P(k)$ mean that Demorgan’s Law can be fully applied to an arbitrarily parenthesized conjunction with $k$ logical statements.

**Base Case:** Prove $P(2)$.

That is prove $\sim (P_1 \land P_2) \leftrightarrow (\sim P_1 \lor \sim P_2)$. However, this is just De Morgan’s Law (one can prove this using truth tables, for instance).

**IH:** Assume $P(k)$ for $k \geq 2$. That is, assume that $P$ is a conjunction of $k$ logical statements which are arbitrarily and fully parenthesized. Then, one can apply the generalized De Morgan’s Law on $\sim P$.

**Ind. Case:** Prove $P(k+1)$ That is, prove: that if $P$ is a conjunction of $k + 1$ logical statements which are arbitrarily and fully parenthesized. Then, one can apply the generalized De Morgan’s Law on $\sim P$.

Since $P$ contains $k + 1$ variables, fully parenthesized, we can write it as $(P' \land P'')$. Apply De Morgan’s on this to get $(\sim P' \lor \sim P'')$. Both $P'$ and $P''$ have $k$ logical statements or fewer, so the inductive hypothesis can apply (that is, we can apply the generalized De Morgan’s to $P'$ and $P''$ to get a disjunction).

5. Let $P(k)$ mean $5 \mid 11^k - 6$.

**Base Case:** Prove $P(1)$.
That is, prove: $5 \div 11^1 - 6$. This is just $5 \div 5$. True, since there exists an integer $m$ (namely $m = 1$) such that $5 = 5m$.

**IH**: Assume $P(k)$ for $k \geq 1$.

That is, assume $5 \div 11^k - 6$. This means there exists a constant, call it $j$, such that $5j = 11^k - 6$. We can rewrite this as: $5j + 6 = 11^k$ (by adding 6 on both sides).

**Ind. Case**: Prove $P(k + 1)$ for $k \geq 1$.

That is, prove $5 \div 11^{k+1} - 6$.

Using the definition of divides, we must find a constant $m$ such that $5m = 11^{k+1} - 6$. This can be rewritten as $5m = 11 \times 11^k - 6$. Apply the IH to $11^k$ to get $5m = 11 \times (5j + 6) - 6$. Then, multiply to get $5m = 55j + 66 - 6$. Simplify to get $5m = 55j + 60$. Factor out a 5 from the RHS to get $5m = 5(11j + 12)$. So, $m = 11j + 12$. Since $j$ is an integer, then $m$ must be as well (by closure of integers under multiplication and addition).

6. Let $P(k)$ mean $a_k \leq 30k$.

**Base Case**: Prove $P(0)$.

That is, prove $a_0 \leq 30 \cdot 0$. Since $a_0 = 0$, this means $0 \leq 30 \cdot 0$, which it is.

**IH**: Assume $P(i)$ for $0 \leq i < k$.

That is, assume $a_{i-1} \leq 30(i - 1)$ for $0 \leq i < k$.

**Ind. Case**: Prove $P(k)$.

That is, prove $a_k \leq 30k$.

\[
\begin{align*}
as_k &= a_{\lfloor \frac{k}{3} \rfloor} + 2a_{\lfloor \frac{k}{4} \rfloor} + 5k & \text{(Def'k of } a_k) \\
&\leq 30\lfloor \frac{k}{3} \rfloor + 2(30\lfloor \frac{k}{4} \rfloor) + 5k & \text{(by IH)} \\
&\leq 30 \frac{k}{3} + 2(30 \frac{k}{4}) + 5k & \text{(since } \lfloor x \rfloor \leq x) \\
&\leq 10k + 15k + 5k & \text{(Simplify)} \\
&\leq 30k & \text{(Simplify)}
\end{align*}
\]

7.

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Assuming player 1 and 2 both play to win, and player 1 moves first. Then, player 1 wins if $n \equiv 1, 3, 4, 5, 6, 7 \pmod{8}$ where $n$ is the number of stones initially. Player 1 loses if there are initially $n \equiv 0, 2 \pmod{8}$ stones.