CMSC 250 Final Exam Saturday, May 18, 2002

Write all answers legibly on the paper provided. If you need extra paper, raise your hand and request a blank paper – you must put your name on and hand-in any paper you receive. The number of points possible for each question is indicated in square brackets – the total number of points on the exam is 200, and you will have exactly 2 hours to complete this exam. You may not use calculators, textbooks or any other aids during this exam.

1. [15 pts.] For each of the following English sentences, translate the meaning into formal notation using the logic symbols (exists, ∀, ∧, ∨, ¬, and →). In addition to these, you may also use mathematical, grouping and set notations symbols as needed. On the next line write the negation of the original statement using formal notation.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Negation</th>
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<tbody>
<tr>
<td>For every pair of positive integers, there is exactly one real number equal to the first of those integers divided by the second. Domain: R = {all reals} and Z⁺ = {all positive integers} Predicate: L(x,y,z) = “x equals (y divided by z)”</td>
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<tr>
<td>For every even integer, there is an integer whose value is exactly half of that integer. Domains: Z = {all integers} and Z_even = {all even integers} Predicate: H(a,b) = “a is equal to half of b”</td>
<td></td>
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<tr>
<td>For every building which is a house there is a person who owns it. Domains: B = {all buildings} and P = {all people} Predicate: H(a) = “a is a house” and O(a,b) = “a is owned by b”</td>
<td></td>
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</tbody>
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**** This area is for grading purposes (points lost per page)- Do not write below this line ****
2. [20 pts.] Use only those rules given on the "cheatsheet" to prove that the following is a valid argument. It is a Valid Argument - you only need to prove that it is.

<table>
<thead>
<tr>
<th></th>
<th>Statement</th>
<th>Reason</th>
<th>Line #s</th>
</tr>
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<tbody>
<tr>
<td>P1</td>
<td>( \forall x \in D, \sim (P(x) \lor R(x)) )</td>
<td></td>
<td></td>
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<tr>
<td>P2</td>
<td>( \exists y \in D, \ M(y) \rightarrow R(y) )</td>
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<tr>
<td>P3</td>
<td>( \forall z \in D, \ Q(z) \rightarrow T(z) )</td>
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<tr>
<td>P4</td>
<td>( \forall w \in D, \sim T(w) \lor P(w) )</td>
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<tr>
<td></td>
<td><strong>Therefore ( \sim \forall x \in D, \ Q(x) \lor M(x) )</strong></td>
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</table>
3. [20 pts.] Assuming you know the following facts:

- \( a_1 = 4, a_2 = 8, a_3 = 14 \) and \( a_4 = 24 \)
- \( \forall n \in \mathbb{Z}^{\geq 5}, a_n = 4a_{n-4} + 3a_{n-2} \)

Prove that \( \forall n \in \mathbb{Z}^{\geq 1}, a_n \geq 2^n \)
4. [20 pts.] Assume you are dealing with a group of 21 people. 8 of the people are female and 13 are male. 5 of the people are children (under 18), 10 of the people are “seniors” (65 years of age or older) and the rest are regular adults.

Answer the following questions about those students. You do not have to do the arithmetic, but you do have to get each answer into a form that includes factorials and/or exponents:

a. How many ways can I line them up to follow one another in one single file line assuming neither a child nor a senior can lead the group?

b. How many ways can I form a single 5 member team assuming the team can not be entirely made of children?

c. If I were to select one name randomly, what is the probability that the name chosen will be a child (assuming no two people in the group share the same name)?

d. How many ways can I make a single 10 member team assuming all of the members of the team must be of the same gender?

e. How many ways can I make a 3 member team assuming the team must be one child, one senior and one regular adult?
5. [20 pts.] Use induction on $n$ to prove that $6((2^{3n+2} + 4 \cdot 5^{n+1})$ for all $n \geq 0$. 
6. [20 pts.] Let \( a, b, c, m, n \in \mathbb{Z}^+ \) such that \( a^m = b^n = c \). Let \( p \in \{\text{primes}\} \) such that \( p | c \). Prove that \( p^{\max(m,n)} | c \).
7. [25 pnts.] Determine the number of elements in $B$ which must map to one single element in $D$ assuming the following definitions for the sets involved.

- Let $A$ be a nonempty finite set with 3 elements. \[ n(A) = 3 \]

- Let $B$ be the set of all binary relations on $A$. \[ n(B) = \]

- Let $P(A)$ indicate the powerset of $A$. \[ n(P(A)) = \]

- Let $C$ be the set strings of length two which are created from the alphabet formed by the elements of $A$. \[ n(C) = \]

- Let $D$ be the cartesian product of $P(A) \times C$. \[ n(D) = \]

- Let $f$ represent any function $f : B \rightarrow D$. \[ \text{number of elements in } B \text{ which must map to one single element in } D = \]

- Prove your last answer from the list above:
8. [20 pts.] Let $A = \{a, b, c, d, e, f\}$. Let $P$ be a partial order relation on $A$ defined by:

$$P = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (f, a), (f, b), (f, c), (f, d), (f, e), (a, c), (e, c), (e, d), (b, d)\}$$

a. Draw the directed graph for $P$.

b. Draw the Hasse diagram for $P$.

c. List all of the least, greatest, maximal, and minimal elements of $P$. If no elements of that type exist, write NONE next to that word.
9. [20 pts.] Either give a counter example to disprove (giving specific members for the sets A, B, C and D) or use only the rules provided on (either side of) the “cheat sheet” along with any definitions from the textbook and/or class to prove the following statement.

Be sure to give the name of the reason which justifies each step you give.

Let \( A, B, C, D \) be sets. \((A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)\)
10. [20 pts.] Let \( M = \{1, 2, \cdots, m\} \) where \( m \geq 1 \), let \( N = \{1, 2, \cdots, n\} \) where \( n \geq 1 \), and let \( E \) be the set of edges in the complete bipartite graph between \( M \) and \( N \). Let \( R \) be a binary relation on \( E \) defined by \( e_iRe_j \leftrightarrow [ (m_i - n_i \equiv m_j - n_j) \mod 3 ] \), where \( e_k \in E \) is defined by its endpoints \( m_k \in M \) and \( n_k \in N \).

a. How many elements are in \( E \)?

b. Prove that \( R \) is an equivalence relation.

c. Let \( M = N = \{1, 2, 3, 4\} \). List the elements of each of the equivalence classes of \( R \).