Due Wednesday, November 5 at the beginning of your discussion section.

You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.

1. Prove for all sets $A$, $B$, and $C$, $[C \subseteq A] \rightarrow [(A \cap B) \times C \subseteq (A \times A) \cap (B \times A)]$.

   **Answer:** Let $A$, $B$, and $C$ be arbitrary sets; assume $C \subseteq A$.
   Let $(x, y) \in (A \cap B) \times C$ be arbitrary.
   $x \in A \cap B \wedge y \in C$ by the definition of cross product.
   $x \in A \wedge x \in B \wedge y \in C$ by the definition of intersection.
   Since $y \in C$, $y \in A$ by the definition of subset.
   Since $x \in A$ and $y \in A$, $x \in A \wedge y \in A$ by conjunctive addition.
   $(x, y) \in A \times A$ by the definition of cross product.
   Since $x \in B$ and $y \in A$, $x \in B \wedge y \in A$ by conjunctive addition.
   $(x, y) \in B \times A$ by the definition of cross product.
   $(x, y) \in A \times A \wedge (x, y) \in B \times A$ by conjunctive addition.
   $(x, y) \in (A \times A) \cap (B \times A)$ by definition of intersection.
   Therefore, for any sets $A$, $B$, and $C$, $[C \subseteq A] \rightarrow [(A \cap B) \times C \subseteq (A \times A) \cap (B \times A)]$ by closing the conditional world and generalizing from the generic particular.

2. Prove for all sets $A$, $B$, and $C$, $(A - B) - C = A - (B \cup C)$.

   **Answer:** Let $A$, $B$, and $C$ be arbitrary sets.
   
   $$(A - B) - C = (A \cap B^c) - C \quad \text{definition of set difference}$$
   $$= (A \cap B^c) \cap C^c \quad \text{definition of set difference}$$
   $$= A \cap (B^c \cap C^c) \quad \text{associativity}$$
   $$= A \cap (B \cup C)^c \quad \text{DeMorgan's Law}$$
   $$= A - (B \cup C) \quad \text{definition of set difference}$$

   Therefore, for all sets $A$ and $B$, $(A - B) - C = A - (B \cup C)$.

3. Prove for all sets $A$, $B$, and $C$, if $C \subseteq B - A$, then $A \cap C = \emptyset$.

   **Answer:** Let $A$, $B$, and $C$ be arbitrary sets; assume $C \subseteq B - A$.
   Therefore, $C \subseteq (B \cap A^c)$ by definition of set difference.
   Since $C \subseteq B \cap A^c$, $\forall p \in C \rightarrow (p \in B \cap A^c)$.
   Let $x$ be some arbitrary element of $A \cap C$ (to show $A \cap C$ is empty, we assume it has some element $x$, and derive a contradiction).
   So $x \in A \cap C$.
   $x \in A \wedge x \in C$ by the definition of intersection.
   $x \in A$ by conjunctive simplification.
   $x \in B^c \vee x \in A$ by disjunctive addition.
   $x \in (B^c \cup A)$ by definition of union.
   $x \notin (B^c \cup A)^c$ by definition of $\notin$. 
\[ x \notin (B \cap A^c) \text{ by DeMorgan’s Law.} \]
\[ x \notin C \text{ by universal modus tollens.} \]

Now we have a contradiction, since \( x \in C \) and \( x \notin C \), which means our assumption that \( x \in A \cap C \) was false.

So \( x \notin A \cap C \) by closing the conditional world with a contradiction.

So \( \forall p \ p \notin A \cap C \) by generalizing from the generic particular.

So \( A \cap C = \emptyset \) by the definition of \( \emptyset \).

4. Prove for all sets \( A \) and \( B \), \( \mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B) \). Explain why these two sets are not necessarily equal.

**Answer:** Let \( A \) and \( B \) be arbitrary sets.

Let \( X \) be an arbitrary element of \( \mathcal{P}(A) \cup \mathcal{P}(B) \).

\[ X \in \mathcal{P}(A) \lor X \in \mathcal{P}(B) \text{ by definition of union.} \]

**Case 1:** Assume \( X \in \mathcal{P}(A) \).

\( X \subseteq A \) by definition of membership in a power set.

Let \( y \) be an arbitrary element of \( X \).

Since \( y \in X \) and \( X \subseteq A \), \( y \in A \) by the definition of subset.

\( y \in A \lor y \in B \) by disjunctive addition.

\( y \in A \cup B \) by definition of union.

\[ \forall y \ y \in X \rightarrow x \in A \cup B \text{ by closing the conditional world.} \]

\( X \subseteq A \cup B \) by the definition of subset.

**Case 2:** Assume \( X \in \mathcal{P}(B) \).

\( X \subseteq B \) by definition of membership in a power set.

Let \( y \) be an arbitrary element of \( X \).

Since \( y \in X \) and \( X \subseteq B \), \( y \in B \) by the definition of subset.

\( y \in A \lor y \in B \) by disjunctive addition.

\( y \in A \cup B \) by definition of union.

\[ \forall y \ y \in X \rightarrow x \in A \cup B \text{ by closing the conditional world.} \]

\( X \subseteq A \cup B \) by the definition of subset.

Since one of the two cases always applies, and both cases lead to the same result, we can conclude that

\( X \subseteq A \cup B \) by the dilemma rule.

\( X \in \mathcal{P}(A \cup B) \) by definition of membership in a power set.

\[ \forall X \ X \in \mathcal{P}(A) \cup \mathcal{P}(B) \rightarrow X \in \mathcal{P}(A \cup B) \text{ by closing the conditional world and generalizing from the generic particular.} \]

\[ \mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B) \text{ by definition of subset.} \]

Therefore, for all sets \( A \) and \( B \), \( \mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B) \) by generalizing from the generic particular.

These sets are not necessarily equal; this can be illustrated if we think about the size of each set. If a set \( A \) has size \( n \), then \( \mathcal{P}(A) \) has size \( 2^n \). If \( A \) and \( B \) have sizes \( x \) and \( y \) respectively, the size of \( A \cup B \) could be as large as \( x + y \). But in general, \( 2^x + 2^y \neq 2^{x+y} \), and two sets with different sizes cannot be equal. This can be illustrated with an example as follows:

Let \( A = \{1\} \) and \( B = \{2\} \). Then \( \mathcal{P}(A) = \{\emptyset, \{1\}\} \), and \( \mathcal{P}(B) = \{\emptyset, \{2\}\} \), and so \( \mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\} \). On the other hand, \( A \cup B = \{1, 2\} \), which means \( \mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \).
5. Let \( n \) and \( k \) be positive integers, and let \( \Sigma \) be an alphabet of size \( n \). What is the the size of the set \( \Sigma^k \)?

**Answer:** \( \Sigma^k \) represents the set of all strings over \( \Sigma \) of length \( k \). Since there are \( n \) letters in our alphabet, there are \( n \) choices for each letter of each string in the set. Since the strings are of length \( k \), \( n(\Sigma^k) = n^k \).

6. Let \( A_1, A_2, \ldots \) be sets. Prove the generalized DeMorgan’s law:

\[
\forall n \in \mathbb{Z}^+ \ (A_1 \cap A_2 \cap \cdots \cap A_n)^c = A_1^c \cup A_2^c \cup \cdots \cup A_n^c
\]

**Hint:** Use induction, and the fact that \( A_1 \cap A_2 \cap \cdots \cap A_n = A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap A_n \).

**Answer:**

**Base Case:** \( (n = 1) \)

\( A_1^c = A_1^c \checkmark \)

**Inductive Hypothesis:** \( (n = k) \)

\( (A_1 \cap A_2 \cap \cdots \cap A_k)^c = A_1^c \cup A_2^c \cup \cdots \cup A_k^c \)

**Inductive Step:** \( (n = k + 1) \)

**Show:** \( (A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1})^c = A_1^c \cup A_2^c \cup \cdots \cup A_k^c \cup A_{k+1}^c \)

**Proof:**

Let \( S = A_1 \cap A_2 \cap \cdots \cap A_k \).

By the IH, \( S^c = (A_1 \cap A_2 \cap \cdots \cap A_k)^c = A_1^c \cup A_2^c \cup \cdots \cup A_k^c \).

\[
(A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1})^c = (S \cap A_{k+1})^c \quad \text{substitution}
\]

\[
= (S^c \cup A_{k+1}^c) \quad \text{DeMorgan’s Law}
\]

\[
= (A_1^c \cup A_2^c \cup \cdots \cup A_k^c \cup A_{k+1}^c) \quad \text{substitution}
\]

And this is what we wanted to show.

7. Given any integer \( k \), explain how to construct \( k \) infinite sets that form a partition of \( \mathbb{Z} \). **Hint:** There is a theorem we studied in number theory that can help.

**Answer:** We can use the quotient-remainder theorem to help us. This theorem says that given a positive integer \( d \) and any integer \( n \), there is a unique integer \( r \), \( 0 \leq r < d \) such that \( n = dq + r \). Let us invoke this theorem with \( d = k \). Since \( r \) can take on \( k \) values (since \( 0 \leq r < d = k \)), we can partition the set \( \mathbb{Z} \) into sets based on the value of \( r \), since \( r \) is unique for each integer. Each set is infinite because if some integer \( n_1 \) has an \( r \)-value of \( r_1 \), then the integer \( n_1 + k \) also has this same \( r \)-value.

These \( k \) sets form a partition since each integer has a unique \( r \)-value for \( r \), so each integer falls into exactly one of the sets.