Predicate Calculus

- Subject / Predicate
  - John / went to the store.
  - The sky / is blue.
- Propositional Logic - uses statements
- Predicate Calculus - uses predicates
  - predicates must be applied to a subject in order to be true or false
- \( P(x) \)
  - means this predicate represented by \( P \)
  - applied to the object represented by \( x \)

Quantification

- \( \exists x \) There exists an \( x \)
- \( \forall x \) For all \( x \)'s
- Domain - set where these subjects come from

- \( \exists x \in \mathbb{Z} \) There exists an \( x \) in the integers
- \( \forall x \in \mathbb{R} \) For all \( x \)'s in the reals

Translation

- A student of mine is wearing a blue shirt.
  - Domain: people who are my students \( S \)
  - Quantification: There is at least one
  - Predicate: wearing a blue shirt
  \( \exists x \in S \) such that \( B(x) \)
  where \( B(x) \) represents "wearing a blue shirt"

- My students are in class.
  - Domain: people who are my students \( S \)
  - Quantification: All of them
  - Predicate: are in class
  \( \forall x \in S \) such that \( C(x) \)
  where \( C(x) \) represents "being in class"
**Negation of Quantified Statements**

~ (Ǝx ∈ people such that H(x))

≡ ∀x ∈ people such that ~ H(x)

~~(There is a person who is here.)~~

For all people, each person is not here.

same in meaning as "There is no person here."

~ (∀ x ∈ people such that H(x))

≡ ∃ x ∈ people such that ~ H(x)

~~(For all people, each person is here.)~~

There is at least one person who is not here.

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**Multiple Predicate Translation**

- **A student of mine is wearing a blue shirt.**
  - Domain: all people  P
  - Quantification: There is at least one
  - Predicates: "wearing a blue shirt" and "is my student"
  
  Ǝx ∈ P such that B(x) ∧ S(x)
  
  B(x) represents "wearing a blue shirt"
  S(x) represents "being my student"

- **My students are in class.**
  - Domain: all people  P
  - Quantification: All of them
  - Predicates: "are in class" and "is my student"
  
  ∀x ∈ P such that S(x) → C(x)
  
  C(x) represents "being in class"
  S(x) represents "being my student"

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**Multiple Quantification**

- ∃y Ǝx S(x,y)
  
  where x ∈ {all chairs} and y ∈ {all people}

- ∀x  ∀y S(x,y)
  
  where x ∈ {all chairs} and y ∈ {all people}

S(x,y) represents "y sitting in x"
Mixed Multiple Quantification
• \( \forall x \in C \ \exists y \in P, \ S(x, y) \)
• \( \forall y \in P \ \exists x \in C, \ S(x, y) \)
• \( \exists y \in P \ \forall x \in C, \ S(x, y) \)
• \( \exists x \in C \ \forall y \in P, \ S(x, y) \)
where \( C = \{ \text{all chairs} \} \) and \( P = \{ \text{all people} \} \)
\( S(x, y) \) represents "y sitting in x"

Negations of Multiply Quantified Statements
• \( \sim(\forall x \in C \ \exists y \in P, \ S(x, y)) \)
• \( \exists x \in C \ \forall y \in P, \ \sim S(x, y) \)
where \( x \in \text{all chairs} \) and \( y \in \text{all people} \)
\( S(x, y) \) represents "y sitting in x"

Other Variations
• Exactly one child attends school
Other Variations

• Exactly one child attends school
  - $\exists c \exists b S A(c,s) \land [\exists p c \exists b e S, p \land c \land A(p,b)]$
  - $\exists c \exists b S, A(c,s) \land [\forall p c \forall b e S, p \land c \land A(p,b)]$

• At most 1 child attends school
  - $\forall c,p \in C \forall s,b \in S, (A(c,s) \land A(p,b)) \rightarrow c=p$
Other Variations

• Exactly one child attends school
  \[ \exists c \in C \ \exists s \in S \ A(c,s) \land \neg[\exists p \in C \ \exists b \in S \ p \neq c \land A(p,b)] \]
  \[ \exists c \in C \ \exists s \in S, A(c,s) \land \neg[\forall p \in C \ \forall b \in S \ p \neq c \land A(p,b)] \]
• At most 1 child attends school
  \[ \forall c,p \in C \forall s,b \in S, (A(c,s) \land A(p,b)) \rightarrow c=p \]
• At least 2 children attend school

Degenerate or Vacuous Cases

• \( \forall s \ B(s) \) - all my students are wearing blue
  \( B(s) \) "student s is wearing blue"
• \( \forall s \ \forall c \ I(s,c) \)
• \( \forall s \ \exists c \ I(s,c) \)
• \( \exists c \ \forall s \ I(s,c) \)
  \( I(s,c) \) "student s is in class c"

If there are no students…
Variants of Quantified Conditional Statements

- **Statement:** $\forall x \in D, P(x) \rightarrow Q(x)$
- **Contrapositive:** $\forall x \in D, \neg Q(x) \rightarrow \neg P(x)$
- **Converse:** $\forall x \in D, Q(x) \rightarrow P(x)$
- **Inverse:** $\forall x \in D, \neg P(x) \rightarrow \neg Q(x)$

- Also applies to Existentially Quantified Conditional Statements

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Euler Diagrams

- Circles used to tell “Truth Sets” for the predicate
  - Where the predicate applied to an object is true
- A dot is used to tell a specific instance
- If “all” then a completely contained circle
- If “some” then an overlapping circle

All college students are brilliant.  
All brilliant people are scientists.  
\[ \therefore \text{All college students are scientists.} \]

Some poets are unsuccessful.  
Some athletes are unsuccessful.  
\[ \therefore \text{Some poets are athletes.} \]

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Rules of Inference for Quantified Statements

<table>
<thead>
<tr>
<th>Universal Modus Ponens</th>
<th>Universal Modus Tollens</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \in D, P(x) \rightarrow Q(x)$</td>
<td>$\forall x \in D, P(x) \rightarrow Q(x)$</td>
</tr>
<tr>
<td>$P(a)$</td>
<td>$\neg Q(a)$</td>
</tr>
<tr>
<td>$a \in D$</td>
<td>$a \in D$</td>
</tr>
<tr>
<td>$\therefore Q(a)$</td>
<td>$\therefore \neg P(a)$</td>
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<tr>
<th>Universal Instantiation</th>
<th>Existential Generalization</th>
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</thead>
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<tr>
<td>$\forall x \in D, P(x)$</td>
<td>$\exists x \in D, P(x)$</td>
</tr>
<tr>
<td>$a \in D$</td>
<td>$c \in D$</td>
</tr>
<tr>
<td>$\therefore P(a)$</td>
<td>$\therefore \exists x \in D, P(x)$</td>
</tr>
</tbody>
</table>
Rules that DON'T exist or need more clarification

• Existential Modus Ponens - Doesn't exist
• Existential Modus Tollens - Doesn't exist

• Universal Generalization: $P(a) \vdash \forall x \in D, P(x)$
  – only if a is completely arbitrary in the domain
• Existential Instantiation: $\exists x \in D, P(x) \vdash P(a)$
  – only if a is completely arbitrary in the domain

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Errors in Deduction

<table>
<thead>
<tr>
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<td>$\forall x \in D, P(x) \to Q(x)$</td>
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<tr>
<td>$Q(a)$</td>
</tr>
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<td>$\therefore P(a)$</td>
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<td>Called: Asserting the Consequence</td>
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<tr>
<td>Called: Denying the Hypothesis</td>
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Easy Formal Direct Proofs by Deduction

• $\forall x \in D, P(x) \to Q(x)$
• $\neg Q(a)$ where $a \in D$
• therefore: $\exists x \in D, \neg P(x)$

• $\forall x \in D, P(x) \to Q(x)$
• $\forall x \in D, R(x) \to \neg P(x)$
• $P(b)$ where $b \in D$
• therefore: $Q(b) \land \neg R(b)$
More Formal Direct Proofs by Deduction

- \( \forall x \in D, P(x) \rightarrow Q(x) \)
- \( \forall x \in D, \neg P(x) \lor R(x) \)
- \( P(b) \) where \( b \in D \)
- therefore: \( \exists x \in D, Q(x) \land R(x) \)

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- \( \forall x \in D, P(x) \rightarrow Q(x) \)
- \( \forall y \in D, \neg P(y) \rightarrow R(y) \)
- \( \exists z \in D, \neg Q(z) \)
- therefore: \( \exists x \in D, R(x) \)

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A(c,s) = "child c attends school s"

- \( \exists c \exists s A(c,s) \)
  - find one child/school combo which makes it true
  - one child attends some school somewhere
- \( \forall c \forall s A(c,s) \)
  - must be true for all child/school combos
  - all children must attend all schools
- \( \forall c \exists s A(c,s) \)
  - for all children select any one school to which that child attends
  - all children attend some school
- \( \forall s \exists c A(c,s) \)
  - for all schools select any one child to which that school attends
  - all schools have at least one child
- \( \exists s \forall c A(c,s) \)
  - select any one school and assert that all children attend that school
  - there is a school that all children attend
- \( \exists c \exists s A(c,s) \)
  - select any one child and assert that all schools are attended by that one child
  - there is a child who attends all schools

---

A(c,s) = "child c attends school s"

- Negation of “Every child attends school”.
  - \( \neg \forall c \exists s A(c,s) \)
- At least one child did not attend school.

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A(c,s) = "child c attends school s"
• Negation of “Every child attends school”.
  – ~[∀c ∃s A(c,s)]
• At least one child did not attend school.
  – ~[∃c ∃s A(c,s)]
    • It is not the case that all children attend school.
  – ∃c ~[∃s A(c,s)]
    • There is one child for whom it is not the case that there exists a school which he/she attends.
  – ∃c ∀s ~A(c,s)
    • There is one child for whom all schools are ones that he/she does not attend.

A(c,s) = "child c attends school s"
• Negation of “At least one child attends school”.
  – ~[∃c ∃s A(c,s)]
• No children attends school.
  – ~[∃c ∃s A(c,s)]
    • It is not the case that there is a child who attends school.
  – ∀c ~[∃s A(c,s)]
    • For all children it is not the case that you can select a school that child attends.
  – ∀c ∀s ~A(c,s)
    • For all child/school combinations it is not the case that the child attends that school.

A(c,s) = "child c attends school s"
• Negation of “At least one child attends school”.
  – ~[∃c ∃s A(c,s)]
• No children attend school.
  – ~[∃c ∃s A(c,s)]
    • It is not the case that there is a child who attends school.
  – ∀c ~[∃s A(c,s)]
    • For all children it is not the case that you can select a school that child attends.
  – ∀c ∀s ~A(c,s)
    • For all child/school combinations it is not the case that the child attends that school.
More Practice in Translation

• No two people share the same toothbrush.

More Practice in Translation

• No two people share the same toothbrush.
  • Development:
    It is not the case (there are two (or more) people sharing a toothbrush).
    ~(there are two (or more) people sharing a toothbrush)
    ~(at least two people share a toothbrush)

  -∃s,m ∈ P ∃t ∈ T, K(s,t) ∨ K(m,t) ∨ s ≠ m

  Another way:
  If we look at every pair of people/toothbrush combination, one of the three is false.

  ∀s,m ∈ P ∀t ∈ T, ~K(s,t) v ~K(m,t) v s=m

More Practice in Translation

• There is a person only a mom could love.
More Practice in Translation

• There is a person only a mom could love.
• Development:
  There is at least one person (only a mom could love).
  There is at least one person (if anyone loves him it must be a mom)
  There is at least one person (if anyone loves him then that person is a mom)

\[ \exists x \in P \forall s \in P, \ L(s, x) \rightarrow M(s) \]

L(s, x) means "s loves x"
M(s) means "s is a mom"

Another way:
There is at least one person, x,(There's nobody in the world who loves x is not a Mom)
There's a person, x, (it's not the case (there is a person who Loves x and is not a Mom)).

\[ \exists x \in P \neg (\exists s \in P, L(s, x) \land \neg M(s)) \]

One More Proof

P1 \( \forall x \in D, [A(x) \lor B(x)] \rightarrow [M(x) \lor N(x)] \)

P2 \( \exists y \in D, A(y) \land \neg N(y) \)

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therefore \( \exists z \in D, M(z) \lor B(z) \)