**Inductive Proofs Must Have**

- **Base Case (value):**
  - where you prove it is true about the base case
- **Inductive Hypothesis (value):**
  - where you state what will be assume in this proof
- **Inductive Step (value):**
  - show:
    - where you state what will be proven below
  - proof:
    - where you prove what is stated in the show portion
    - this proof must use the Inductive Hypothesis sometime during the proof

**Prove this statement:** \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \)

### Base Case (n=1):

\[
\sum_{i=1}^{1} i = \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1
\]

### Inductive Hypothesis (n=p):

\[
\sum_{i=1}^{p} i = \frac{p(p+1)}{2}
\]

### Inductive Step (n=p+1):

**Show:** \( \sum_{i=1}^{p+1} i = \frac{(p+1)((p+1)+1)}{2} \)

**Proof:** (in class)

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**Variations**

- \( 2+4+6+8+\ldots+20 = ?? \)
- If you can use the fact:
  \[
  \sum_{i=1}^{n} i = \frac{n(n+1)}{2}
  \]
  - Rearrange it into a form that works.
  - If you can’t – you must prove it from scratch
Less Mathematical Example

- If all we had was 2 and 5 cent coins, we could make any value greater than 3.
- Base Case (n = 4):
- Inductive Hypothesis (n=k):
- Inductive Step (n=k+1):
  
  show:
  proof:

More Examples
to be done in class

- \( \forall n \in Z_{\geq 1}, 3 \mid (n^3 - n) \)
- \( \sum_{k=0}^{n} 2^k = 2^{n+1} - 1 \)
- Geometric Progression

Proving Inequalities with Induction

- Inductive Hypothesis
  - has the form \( y < z \)
- Inductive Step
  - needs to prove something of the form \( x < z \)
- Two methods for the proof part
  - use whichever you like
  - transitivity
    - find a value between (b)
    - prove that \( b < z \)
    - prove that \( x < b \)
  - book method
    - Substitute “unequals” as long as the signs don’t change
    - Add unequals to unequals as long as always adding correct sides
Prove this statement:
\[ \forall n \in \mathbb{Z}_{\geq 3}, 2n + 1 \leq 2^n \]

Base Case (n=3):
LHS: 2(3) + 1 = 6 + 1 = 7
RHS: 2^3 = 8
LHS \leq RHS

Inductive Hypothesis (n=k):
\[ 2k + 1 \leq 2^k \]

Inductive Step (n=k+1):
Show:
\[ 2(k + 1) + 1 \leq 2^{k+1} \]
Proof: (both methods done in class)

Another Example
with inequalities
\[ \forall n \in \mathbb{Z}_{\geq 2}, \forall x \in \mathbb{Z}^{>-1}, 1 + nx \leq (1 + x)^n \]

Strong Induction
• Implication changes slightly
  – if true for all lesser elements, then true for current
• P(i) \( \forall i \in \mathbb{Z} \ a \leq i < k \rightarrow P(k) \)
• P(i) \( \forall i \in \mathbb{Z} \ a \leq i \leq k \rightarrow P(k+1) \)

Regular Induction
• P(k) \( \rightarrow P(k+1) \)
• P(k-1) \( \rightarrow P(k) \)
Recurrence Relation Example

- Assume the following definition of a function:

\[ a_1 = 1 \]
\[ a_2 = 3 \]
\[ \forall k \in \mathbb{Z}^{\geq 3}, a_k = a_{k-1} + 2a_{k-2} \]

- Prove the following definition property:

\[ \forall n \in \mathbb{Z}^{\geq 1}, a_n \in \mathbb{Z}^{odd} \]

All Integers greater than 1 are divisible by a prime

**Base Case (n=2):**

\[ 2|2 \quad \exists p \in \mathbb{Z}_{prim} \]

**Inductive Hypothesis (n=i \( \forall i \leq k)\):**

\[ \exists p \in \mathbb{Z}_{prim} \]

**Inductive Step (n=k):**

show: \[ \exists p \in \mathbb{Z}_{prim} \]

proof:

A Factorial Example

\[ \forall n \in \mathbb{Z}^{\geq 2}, \frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2} \]
Another Example

• Assume the following definition of a recurrence relation:
  \[ a_1 = 0 \]
  \[ a_2 = 2 \]
  \[ \forall i \in \mathbb{Z}^{\geq 1}, a_{i+2} = 3a_{i+1} + 2 \]

• Prove that all elements in this relation have this property:
  \[ \forall n \in \mathbb{Z}^{\geq 1}, a_n \in \mathbb{Z}^{\text{even}} \]

Well-Ordering Principle

• For any set S of
  – one or more
  – integers
  – all larger than some value
• S has a least element

Use this to prove the Quotient Remainder Theorem

• The quotient-remainder theorem said
  – Given
    • any positive integer \( n \)
    • and any positive integer \( d \)
  – There exists an \( r \) and a \( q \)
    • where \( n = dq + r \)
    • where \( 0 \leq r < d \)
    • which are integers
    • which are unique
Steps to proving the quotient-remainder theorem

- Define $S$ as the set of all non-negative integers in the form $n - dk$ (all integers $k$)
- Prove that it is non-empty
- Prove that we can apply the Well-Ordering Principle
- Then it has a least element
- Prove that the least element ($r$) is $0 \leq r < d$