Sets

Definition of a Set:
NAME = {list of elements or description of elements}
  i.e.  B = \{1,2,3\}  or  C = \{x \in \mathbb{Z}^+ \mid -4 < x < 4\}

Axiom of Extension:
A set is completely defined by its elements
  i.e.  \{a,b\} = \{b,a\} = \{a,b,a\} = \{a,a,a,b,b,b\}

Subset

A \subseteq B \iff \forall x \in U, x \in A \rightarrow x \in B
  A is contained in B
  B contains A

A \not\subseteq B \iff \exists x \in U, x \in A \land x \notin B

Relationship between membership and subset:
  \forall x \in U, x \in A \leftrightarrow \{x\} \subseteq A

Definition of set equality:  A = B \iff A \subseteq B \land B \subseteq A
Same Set or Not??

\[ X = \{ x \in \mathbb{Z} \mid \exists p \in \mathbb{Z}, x = 2p \} \]
\[ Y = \{ y \in \mathbb{Z} \mid \exists q \in \mathbb{Z}, y = 2q-2 \} \]

\[ A = \{ x \in \mathbb{Z} \mid \exists i \in \mathbb{Z}, x = 2i+1 \} \]
\[ B = \{ x \in \mathbb{Z} \mid \exists i \in \mathbb{Z}, x = 3i+1 \} \]
\[ C = \{ x \in \mathbb{Z} \mid \exists i \in \mathbb{Z}, x = 4i+1 \} \]

Set Operations
Formal Definitions and Venn Diagrams

**Union:**

\[ A \cup B = \{ x \in U \mid x \in A \lor x \in B \} \]

**Intersection:**

\[ A \cap B = \{ x \in U \mid x \in A \land x \in B \} \]

**Complement:**

\[ A^c = A' = \{ x \in U \mid x \notin A \} \]

**Difference:**

\[ A - B = \{ x \in U \mid x \in A \land x \notin B \} \]
\[ A - B = A \cup B' \]
Ordered n-tuple and the Cartesian Product

- **Ordered n-tuple** – takes order and multiplicity into account
- \((x_1, x_2, x_3, \ldots, x_n)\)
  - n values
  - not necessarily distinct
  - in the order given
- \((x_1, x_2, x_3, \ldots, x_n) = (y_1, y_2, y_3, \ldots, y_n) \leftrightarrow \forall i \in \mathbb{Z}^{1 \leq i \leq n}, x_i = y_i\)

- **Cartesian Product**
  \(A \times B = \{(a, b) \mid a \in A \land b \in B\}\)

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Formal Languages

- \(\Sigma = \text{alphabet} = \text{a finite set of symbols}\)
- string over \(\Sigma = \)
  - empty (or null) string denoted as \(\varepsilon\)
  - OR
  - ordered n-tuple of elements
- \(\Sigma^n = \text{set of strings of length n}\)
- \(\Sigma^* = \text{set of all finite length strings}\)
Empty Set Properties

1. $\emptyset$ is a subset of every set.
2. There is only one empty set.
3. The union of any set with $\emptyset$ is that set.
4. The intersection of any set with its own complement is $\emptyset$.
5. The intersection of any set with $\emptyset$ is $\emptyset$.
6. The complement of the universal set is $\emptyset$ and the complement of the empty set is the universal set.

Other Definitions

• Proper Subset

$A \subset B \iff A \subseteq B \land A \neq B$

• Disjoint Set

A and B are disjoint

$\iff$A and B have no elements in common

$\iff \forall x \in U, x \in A \rightarrow x \notin B \land x \in B \rightarrow x \notin A$

$A \cap B = \emptyset \iff$A and B are Disjoint Sets

• Power Set

$\mathcal{P}(A) =$ set of all subsets of A
Properties of Sets in Theorems 5.2.1 & 5.2.2

- Inclusion
  \[ A \cap B \subseteq A \quad A \cap B \subseteq B \]
  \[ A \subseteq A \cup B \quad B \subseteq A \cup B \]

- Transitivity
  \[ A \subseteq B \land B \subseteq C \rightarrow A \subseteq C \]

- DeMorgan’s for Complement
  \[ (A \cup B)' = A' \cap B' \quad (A \cap B)' = A' \cup B' \]

- Distribution of union and intersection
  \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

Using Venn Diagrams to help find counter example

\[ A \cup (B \cap C) = ? = (A \cap B) \cup (A \cap C) \]
\[ A \cup (B \cap C) = ? = (A \cap B) \cap C \]

\[ A \cup (B \cap C) = ? = (A \cap B) \cup (A \cap C) \]
Deriving new Properties using rules and Venn diagrams

\[ B - (A \cap C) = (B - A) \cup (B - C) \]

\[ A - B = A - (A \cap B) \]

\[ A \subseteq B \land A \subseteq C \rightarrow A \subseteq (B \cap C) \]

Partitions of a set

- A collection of nonempty sets \{A_1, A_2, \ldots, A_n\} is a partition of the set A if and only if
  1. \( A = A_1 \cup A_2 \cup \ldots \cup A_n \)
  2. \( A_1, A_2, \ldots, A_n \) are mutually disjoint
Proofs about Power Sets

Power set of $A = \mathcal{P}(A) =$ Set of all subsets of $A$

- Prove that
  \[ \forall A, B \in \{\text{sets}\}, A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B) \]

- Prove that (where $n(X)$ means the size of set $X$)
  \[ \forall A \in \{\text{sets}\}, n(A) = k \rightarrow n(\mathcal{P}(A)) = 2^k \]