Graphs

• Formal Definition of Graph G is 2 finite sets
  – \( V(G) = \text{set of vertices} \) & \( E(G) = \text{set of edges} \)

• Example:
  – \( V(H) = \{a,b,c,d,e\} \)  \( E(H) = \{\{a,c\},\{c,e\},\{e,b\},\{b,d\},\{d,a\}\} \)
  – \( V(K) = \{a,b,c,d\} \)  \( E(K)= \{(a,b),(b,a),(a,d),(d,a),(c,c)\} \)

• Variations
  – digraph : edges are ordered tuples
  – multi-graph : edge list is a multiset (bag) not set
  – simple graph : no parallel edges and no “reflexive loops”
  – connected graph: can get from any vertex to any other
  – complete graph: has an edge for every pair of vertices
  – complete bipartite graph: 2 subsets of vertices (u and v), edge from each v to each u, no edges connecting u elements and no edges connecting v elements

• Subgraph: H is a subgraph of G \( \iff V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \)

Counting in Graphs

• Number of Edges Possible
  – complete (simple) graph \( n(E(K_n)) = \sum_{i=1}^{n-1} i \)
  – complete bipartite graph \( n(E(K_{x,y})) = x * y \)

• Degree of a Vertex
  = number of times that vertex is the endpoint of an edge
  = number of edges incident on it with self-loops counted twice

Isomorphism

• \((G \text{ is isomorphic to } H) \iff \) There exists a bijective function \( f_1:(V(G)) \to V(H) \) and a bijective function \( f_2:(E(G)) \to E(H) \)
Traversing a Graph

<table>
<thead>
<tr>
<th>Name</th>
<th>Repeated Edges</th>
<th>Repeated Verticies</th>
<th>Same end/start</th>
</tr>
</thead>
<tbody>
<tr>
<td>Walk</td>
<td>allowed</td>
<td>allowed</td>
<td>allowed</td>
</tr>
<tr>
<td>Path</td>
<td>NO</td>
<td>allowed</td>
<td>allowed</td>
</tr>
<tr>
<td>Simple Path</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>Closed Walk</td>
<td>allowed</td>
<td>allowed</td>
<td>YES</td>
</tr>
<tr>
<td>Circuit</td>
<td>NO</td>
<td>allowed</td>
<td>YES</td>
</tr>
<tr>
<td>Simple Circuit</td>
<td>NO</td>
<td>only the start/end</td>
<td>YES</td>
</tr>
</tbody>
</table>

Euler Circuit

- A circuit that contains every edge and every vertex
  - starts and stops at the same point
  - uses every vertex at least once
  - uses every edge exactly once
- G has an Euler Circuit $\iff$ G is a connected graph and Every vertex of G has even degree

Hamiltonian Circuit

- A simple circuit that contains every vertex.
  - starts and stops at the same point
  - uses every vertex exactly once (except the first and last)
  - does not repeat an edge