Graphs

- Formal Definition of Graph G is 2 finite sets
  - \( V(G) \) = set of vertices & \( E(G) \) = set of edges

- Example:
  - \( V(H) = \{a,b,c,d,e\} \) \( E(H) = \{\{a,c\},\{c,e\},\{e,b\},\{b,d\},\{d,a\}\} \)
  - \( V(K) = \{a,b,c,d\} \) \( E(K) = \{\{a,b\},\{b,a\},\{a,d\},\{d,a\},\{c,c\}\} \)

- Variations
  - digraph: edges are ordered tuples
  - multi-graph: edge list is a multiset (bag) not set
  - simple graph: no parallel edges and no “reflexive loops”
  - connected graph: can get from any vertex to any other
  - complete graph: has an edge for every pair of vertices
  - complete bipartite graph: 2 subsets of vertices (u and v), edge from each v to each u, no edges connecting u elements and no edges connecting v elements

- Subgraph: \( H \) is a subgraph of \( G \) \( \iff V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \)

Counting in Graphs

- Number of Edges Possible
  - complete (simple) graph
  - complete bipartite graph

- Degree of a Vertex
  - number of times that vertex is the endpoint of an edge
  - number of edges incident on it with self-loops counted twice

Isomorphism

- \((G \text{ is isomorphic to } H) \iff \text{ There exists a bijective function } f_1: (V(G)) \to V(H) \text{ and a bijective function } f_2: (E(G)) \to E(H)\)

Traversing a Graph

<table>
<thead>
<tr>
<th>Name</th>
<th>Repeated Edges</th>
<th>Repeated Vertices</th>
<th>Same end/start</th>
</tr>
</thead>
<tbody>
<tr>
<td>Walk</td>
<td>allowed</td>
<td>allowed</td>
<td>allowed</td>
</tr>
<tr>
<td>Path</td>
<td>NO</td>
<td>allowed</td>
<td>allowed</td>
</tr>
<tr>
<td>Simple Path</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>Closed Walk</td>
<td>allowed</td>
<td>allowed</td>
<td>YES</td>
</tr>
<tr>
<td>Circuit</td>
<td>NO</td>
<td>allowed</td>
<td>YES</td>
</tr>
<tr>
<td>Simple Circuit</td>
<td>NO</td>
<td>only the start/end</td>
<td>YES</td>
</tr>
</tbody>
</table>
Euler Circuit
• A circuit that contains every edge and every vertex
  – starts and stops at the same point
  – uses every vertex at least once
  – uses every edge exactly once
• G has an Euler Circuit $\iff$ G is a connected graph and
  Every vertex of G has even degree

Hamiltonian Circuit
• A simple circuit that contains every vertex.
  – starts and stops at the same point
  – uses every vertex exactly once (except the first and last)
  – does not repeat an edge