Relations

- binary relations \( xRy \)
- on sets \( x \in X, y \in Y \) \( R \subseteq X \times Y \)
- Example:
  "less than" relation from \( A = \{0,1,2\} \) to \( B = \{1,2,3\} \)
  - use traditional notation
    - \( 0 < 1, 0 < 2, 0 < 3, 1 < 2, 1 < 3, 2 < 3 \)
  - use set notation
    - \( A \times B = \{(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\} \)
    - \( R = \{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\} \)
  - or use Arrow Diagrams

Formal Definition

- (binary) relation from \( A \) to \( B \) where \( x \in A, y \in B \) \( (x,y) \in A \times B \) \( xRy \leftrightarrow (x,y) \in R \)
  - finite example: \( A = \{1,2\}, B = \{1,2,3\} \)
  - infinite example: \( A = \mathbb{Z}, B = \mathbb{Z} \)
    - \( aRb \leftrightarrow a-b \in \mathbb{Z}^{\text{even}} \)

Different kinds of Relations

- \( x \in R, y \in R \) \( (x,y) \in C \leftrightarrow x^2 + y^2 = 1 \)
- \( \Sigma = \{0,1\} \) \( \Sigma^3 = \{\text{all strings of length 3}\} \)
  - \( xEy \leftrightarrow x \neq y, \text{but first characters match and last characters match} \)
The Inverse of the Relations and The Complement of the Relation

- \( R : A \rightarrow B \) \( R = \{(x,y) \in A \times B | xRy\} \)
- \( R^{-1} : B \rightarrow A \) \( R^{-1} = \{(y,x) \in B \times A | (x,y) \in R\} \)

\( \forall x \in X \forall y \in Y, (y,x) \in R^{-1} \leftrightarrow (x,y) \in R \)

\( \forall x \in X \forall y \in Y, (x,y) \in R \leftrightarrow (x,y) \notin R \)

- \( D : \{1,2\} \rightarrow \{2,3,4\} \)
  \( D = \{(1,2),(2,3),(2,4)\} \) \( D^{-1} = ? \)
- \( S = \{(x,y) \in R \times R | y = 2x\} \)
  \( S^{-1} = ? \)

Digraph

- When it is one set to itself use a directed graph rather than the arrow diagram
- \( R = \{(x,y) \in A \times A | x \text{ divides } y\} \)
  where \( A = \{2,3,4,5,6\} \)

Properties of Relations

- Reflexive
  \( R \) is Reflexive \( \Leftrightarrow \forall x \in A, xRx \)

- Symmetric
  \( R \) is Symmetric \( \Leftrightarrow \forall x, y \in A, xRy \rightarrow yRx \)

- Transitive
  \( R \) is Transitive \( \Leftrightarrow \forall x, y, z \in A, xRy \wedge yRz \rightarrow xRz \)
Proving Properties on Infinite Sets

- \( \forall m,n \in \mathbb{Z}, \ m \equiv n \)

- Reflexive
- Symmetric
- Transitive

Closures over the Properties

- Reflexive Closure
- Symmetric Closure
- Transitive Closure

- \( R^c \) has property \( x \)
- \( R \subseteq R^c \)
- \( R^c \) is the minimal addition to \( R \)
  
  (if \( S \) is any other transitive relation that contains \( R\), \( R^c \subseteq S\))

Equivalence Relations

- Partition the elements:
  any elements “related” are in the same partition
- Equivalence Relations are
  - Reflexive
  - Symmetric
  - Transitive
- Partitions are called Equivalence Classes
  - \( [a] = \) equivalence class containing \( a \)
  - \( [a] = \{ x \in A \mid xRa \} \)
Examples

- $R: X \to X$ with $X = \{a, b, c, d, e, f\}$
- $\{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f), (a,e),(a,d),(d,a),(d,e),(e,a),(e,d),(b,f),(f,b)\}$
- Lemma 10.3.3 If $A$ is a set and $R$ is an equivalence relation on $A$ and $x$ and $y$ are elements of $A$, then either $[x] \cap [y] = \emptyset$ or $[x] = [y]$

Union, Intersection, Difference and Composition

- $R: A \to B$ and $S: A \to B$
  - $R \cup S = \{(x, y) \in A \times B | x \in R \lor (x, y) \in S\}$
  - $R \cap S = \{(x, y) \in A \times B | x \in R \land (x, y) \in S\}$
  - $R - S = \{(x, y) \in A \times B | x \in R \land (x, y) \notin S\}$
  - $S - R = \{(x, y) \in A \times B | x \in S \land (x, y) \notin R\}$

- $R: A \to B$ and $S: B \to C$
  - $S \circ R = \{(a, c) \in A \times C | \exists b \in B, (a, b) \in R \land (b, c) \in S\}$

Matrix Representation of a Relation

- $R^g = [m_{ij}]$ with $m_{ij} = \begin{cases} 1 & \text{iff } (i,j) \in R \\ 0 & \text{iff } (i,j) \notin R \end{cases}$
- Example:
  - $R: \{1,2,3\} \to \{1,2\}$
  - $R = \{(2,1),(3,1),(3,2)\}$
  - $M_R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$
Powers of Relation

• For relation M,
  – \( M^1 \) = the list of paths available in 1 step
  – \( M^2 \) = the list of paths available in 2 steps
  – \( M^3 \) = the list of paths available in 3 steps
  – …
  – \( M^n \) = the list of paths available in any number of steps
• through composition
• through matrix multiplication

Other Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflexive</td>
<td>( \forall a \in A, aRa )</td>
</tr>
<tr>
<td>Irreflexive</td>
<td>( \forall a \in A, \neg aRa )</td>
</tr>
<tr>
<td>Symmetric</td>
<td>( \forall a, b \in A, aRb \rightarrow bRa )</td>
</tr>
<tr>
<td>Antisymmetric</td>
<td>( \forall a, b \in A, aRb \land bRa \rightarrow a = b )</td>
</tr>
<tr>
<td>Asymmetric</td>
<td>( \forall a, b \in A, aRb \rightarrow bRa )</td>
</tr>
<tr>
<td>Non-symmetric</td>
<td>( \forall a, b \in A, a \neq b \rightarrow (aRb \leftrightarrow bRa) )</td>
</tr>
<tr>
<td>Transitive</td>
<td>( \forall a, b, c \in A, aRb \land bRc \rightarrow aRc )</td>
</tr>
</tbody>
</table>

Partial Order Relation

• R is a Partial Order Relation if and only if
  R is Reflexive, Antisymmetric and Transitive
• Partial Order Set (POSET)
  \((S, R) = R\) is a partial order relation on set S
• Examples
  – \((\mathbb{Z}, \geq)\)
  – \((\mathbb{Z}, |)\)  \{note: | symbolizes divides\}
Total Ordering

- When all pairs from the set are “comparable” it is called a Total Ordering

- $a$ and $b$ are comparable if and only if $a R b$ or $b R a$

- $a$ and $b$ are non-comparable if and only if $a R b$ and $b R a$

Hasse Diagram

1. take the digraph (since it represents the same relation)
2. arrange vertices so all arrows go upward (since it is antisymmetric we know this is possible)
3. remove the reflexive loops (since we know it is reflexive these are not necessary)
4. remove the transitive arrows (since we know it is transitive, these are not necessary)
5. make the remaining edges non-directed (since we know they are all going upward, the direction is not necessary)

Hasse Diagram Example

- The POSET $([1,2,3,9,18],\mid)$
- The POSET $([1,2,3,4],\geq)$
- The POSET $(P\{a,b,c\},\subseteq)$

- Draw complete digraph diagrams of these relations
- Derive Hasse Diagram from those
Terminology

• Maximal : $a \in A$ is maximal
  $\iff \forall b \in A \ (bRa \vee a \text{ and } b \text{ are not comparable})$

• Minimal : $a \in A$ is minimal
  $\iff \forall b \in A \ (aRb \vee a \text{ and } b \text{ are not comparable})$

• Greatest : $a \in A$ is greatest $\iff \forall b \in A \ (bRa)$

• Least : $a \in A$ is least $\iff \forall b \in A \ (aRb)$

Topological Sorting

1. select any minimal
   a) put it into the list
   b) remove it from the Hasse diagram
2. Repeat until all members are in the list

Example: $\{2,3,4,6,18,24\}$