1. [20 pnts.] Disprove by counter example or Prove each of the following:

a. The sum of any rational number and any integer is rational.

show: \( \forall r \in Q, \forall m \in Z, r + m \in Q \)

PROOF:
Let \( r \in Q \) and \( m \in Z \) be arbitrary.

Since \( r \in Q \), \( \exists a, b \in Z \) where \( b \neq 0 \), \( r = \frac{a}{b} \) by definition of rational

\[
r + m = \frac{a}{b} + m
\]

by substitution

\[
= \frac{a}{b} + \frac{mb}{b}
\]

by multiplying second term by \( \frac{b}{b} \)

\[
= \frac{a + mb}{b}
\]

by moving them over the common denominator

\( a + mb \in Z \) because of closure of \( Z \) during the operations of + and *

\( b \in Z \) where \( b \neq 0 \) as defined above

therefore \( r + m \in Q \) because it is a quotient of integers as required by the definition of rational.

\( \forall r \in Q, \forall m \in Z, r + m \in Q \) by generalizing from the Generic Particular.

b. For every integer \( n \), \( n^2 - n + 3 \equiv 1 \)

\( \forall n \in Z, n^2 - n + 3 \equiv 1 \)

PROOF:
Let \( n \) be arbitrary in \( Z \).

By the Quotient Remainder Theorem, we know that \( \exists q \in Z, n = 2q + 0 \vee n = 2q + 1 \)

Case 1 \( (n = 2q + 0) \):

\[
\begin{align*}
n^2 - n + 3 &= (2q)^2 - (2q) + 3 \\
n^2 - n + 3 &= 4q^2 - 2q + 2 + 1 \\
n^2 - n + 3 &= 2(2q^2 - q + 1) + 1
\end{align*}
\]

Since \( 2q^2 - q + 1 \in Z \) by closure of integers in multiplication and addition,

\( \exists j \in Z, n^2 - n + 3 = 2j + 1 \) by substitution.

\( (n^2 - n + 3) - 1 = 2j \) by subtracting 1 from both sides.

\( 2 \mid ((n^2 - n + 3) - 1) \) by definition of divides.

\( n^2 - n + 3 \equiv 1 \) by definition of equivalence in a mod

Case 2 \( (n = 2q + 1) \):

\[
\begin{align*}
n^2 - n + 3 &= (2q + 1)^2 - (2q + 1) + 3 \\
n^2 - n + 3 &= 4q^2 + 4q + 1 - 2q - 1 + 3 \\
n^2 - n + 3 &= 4q^2 + 2q + 2 + 1 \quad \text{by algebra}
\end{align*}
\]
\[n^2 - n + 3 = 2(2q^2 + q + 1) + 1\] by factoring
Since \(2q^2 + q + 1 \in \mathbb{Z}\) by closure of integers in multiplication and addition
\[\exists k \in \mathbb{Z}, n^2 - n + 3 = 2k + 1\] by substitution
\[(n^2 - n + 3) - 1 = 2k\] by subtracting 1 from both sides.
\[2 \mid [(n^2 - n + 3) - 1]\] by definition of divides.
\[n^2 - n + 3 \equiv_2 1\] by definition of equivalence in a mod

Since the quotient remainder theorem tells us that these are the only two possibilities and both of these lead to the fact that \(n^2 - n + 3 \equiv_2 1\), by dilemma we know that \(n^2 - n + 3 \equiv_2 1\)

\[\forall n \in \mathbb{Z}, n^2 - n + 3 \equiv_2 1\] by generalizing from the Generic Particular.

2. [4 pnts.] Write the standard factored form of 1050:
\[1050 = 2^1 \times 3^1 \times 5^2 \times 7^1\]

3. [6 pnts.] Use the unique factorization theorem and suppose that \(m\) is an integer such that
\[5 \times 4 \times 3 \times 2 \times m = 10 \times 11 \times 12 \times 13\]
Circle Yes or No for each of the following: Yes means that this is something that must be true, No means it doesn’t necessarily need to be true:

<table>
<thead>
<tr>
<th></th>
<th>Yes/No</th>
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</thead>
<tbody>
<tr>
<td>a) 10(\mid m)</td>
<td>YES (NO)</td>
</tr>
<tr>
<td>b) 11(\mid m)</td>
<td>(YES) NO</td>
</tr>
<tr>
<td>c) 12(\mid m)</td>
<td>YES (NO)</td>
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<tr>
<td>d) 13(\mid m)</td>
<td>(YES) NO</td>
</tr>
<tr>
<td>e) 24(\mid m)</td>
<td>YES (NO)</td>
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<tr>
<td>f) 143(\mid m)</td>
<td>(YES) NO</td>
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