Password for today: banana
Let $a_0 = 1, \ a_1 = 2$. For $k \geq 2$, let $a_k = a_{k-1} + a_{k-2}$

Claim: For all $n \geq 0$ : $a_n \leq 2^n$

Proof: I will apply strong induction on $n$.
Base Case $(n = 0, 1)$: $a_0 = 1 \leq 2^0, \ a_1 = 2 \leq 2^1$.
Inductive Hypothesis: Let $k \geq 1$. Assume (for all $i \leq k$) that $a_i \leq 2^i$
Inductive Step: [I must show $a_{k+1} \leq 2^{k+1}$]

$a_{k+1} = \text{[From defn of sequence]}

a_k + a_{k-1} \leq \text{[By I.H.]}

2^k + 2^{k-1} < \text{[Noting that $2^{k-1} < 2^k$]}

2^k + 2^k =

2(2^k) =

2^{k+1}
Let $a_0 = 0$, $a_1 = 7$. For $k \geq 2$, let $a_k = 2a_{k-1} + 3a_{k-2}$

Claim: For all $n \geq 0$: $a_n \equiv_7 0$

Proof: I will apply strong induction on $n$.
Base Case ($n = 0, 1$): $a_0 = 0$, $a_1 = 7$. These are obviously congruent to 0 (mod 7).
Inductive Hypothesis: Let $k \geq 1$. Assume (for all $i \leq k$) that $a_i \equiv_7 0$
Inductive Step: [I must show $a_{k+1} \equiv_7 0$]

$a_{k+1} = [\text{From defn of sequence}]
2a_k + 3a_{k-1}$.
By the I.H., both $a_k$ and $a_{k-1}$ are divisible by 7, so $a_k = 7r$ and $a_{k-1} = 7s$ for some integers $r, s$.
$a_{k+1} = 2(7r) + 3(7s) = 7(2r + 3s)$.
Noting that $2r + 3s$ is an integer (the integers are closed under multiplication and addition), we see that $a_{k+1}$ is a multiple of 7, hence $a_{k+1} \equiv_7 0$. 

Let $a_0 = 0, \ a_1 = 4$. For $k \geq 2$, let $a_k = 6a_{k-1} - 5a_{k-2}$

Claim: For all $n \geq 0$: $a_n = 5^n - 1$

Proof: I will apply strong induction on $n$.
Base Case ($n = 0, 1$): $a_0 = 0 = 5^0 - 1, \ a_1 = 4 = 5^1 - 1$.
Inductive Hypothesis: Let $k \geq 1$. Assume (for all $i \leq k$) that $a_i = 5^i - 1$
Inductive Step: [I must show $a_{k+1} = 5^{k+1} - 1$]

$a_{k+1} = [\text{From defn of sequence}]
6a_k - 5a_{k-1} = [\text{By I.H.}]
6(5^k - 1) - 5(5^{k-1} - 1) =
6(5^k) - 6 - 5(5^{k-1}) + 5 =
6(5^k) - 5^k - 1 =
5(5^k) - 1 =
5^{k+1} - 1
Claim: For all \( n \geq 2 \): \( n \) can be expressed as a product of primes.

Proof: I will apply strong induction on \( n \).
Base Case (\( n = 2 \)): \( 2 \) is itself, prime.
Inductive Hypothesis: Let \( k \geq 2 \). Assume (for all \( 2 \leq i \leq k \)) that \( i \) is a product of primes.
Inductive Step: [I must show \( k + 1 \) is a product of primes.]
Case \( k + 1 \) is prime: \( k + 1 \) is its own prime factorization.
Case \( k + 1 \) is composite:
\( k + 1 = ab \) for some natural numbers \( a \) and \( b \) such that \( 2 \leq a \leq k \) and \( 2 \leq b \leq k \). The Induction Hypothesis applies to both \( a \) and \( b \), so both \( a \) and \( b \) can be expressed as a product of primes, hence their product \((k + 1)\) can be expressed as a product of primes.
Claim: For $n \geq 1$: It takes exactly $n - 1$ breaks to divide a chocolate bar of size $n$ into individual squares.

Proof: I will apply strong induction on $n$.

Base Case ($n = 1$): There is nothing to do, so it takes 0 breaks.

Inductive Hypothesis: Let $k \geq 1$. Assume (for all $1 \leq i \leq k$) that it takes $i - 1$ breaks to divide a bar of size $i$.

Inductive Step: [I must show it takes $k$ breaks to divide a bar of size $k + 1$]

Suppose you break a bar of size $k + 1$ into two pieces. If the first piece is size $a$ then the size of the second piece is $k + 1 - a$.

The Induction Hypothesis applies to both of those pieces, so the number of breaks to finish each piece will be $a - 1$ and $(k + 1 - a) - 1$, respectively. So to break the original bar of size $k$ into squares it takes 1 initial break plus $a - 1$ breaks to finish the first piece plus $(k + 1 - a) - 1$ breaks to finish the other piece:

$$1 + (a - 1) + (k + 1 - a - 1) = k.$$