Claim: \((\forall x, y \in \mathbb{Z})[x^2 - 4y \neq 2]\)

Proof:

Let \(a, b \in \mathbb{Z}\), selected arbitrarily.
Suppose (BWOC) that \(a^2 - 4b = 2\). [1]
\[a^2 = 4b + 2 = 2(2b + 1)\]
Since \(\mathbb{Z}\) is closed under addition, \((2b + 1) \in \mathbb{Z}\), so \(a^2\) is even.
By our Lemma, that means \(a\) is even, so \(a = 2k\) (some \(k \in \mathbb{Z}\)).
Substituting into [1], we get:
\[(2k)^2 - 4b = 2\]
\[4k^2 - 4b = 2\]
\[4(k^2 - b) = 2\]
\[(k^2 - b) = 1/2.\]
But now the left side is an integer (since \(\mathbb{Z}\) is closed under multiplication and subtraction), but the right side is not. \(\times\)

Since \(a, b\) were selected arbitrarily, the proposition holds for all integers. \(\square\)
Claim: \( \sqrt{2} \) is irrational.

Proof:

Suppose (BWOC) that \( \sqrt{2} \) were rational. 
\( \sqrt{2} = a/b \) for some \( a, b \in \mathbb{Z} \), with \( b \neq 0 \)

We may assume (WLOG) that \( a \) and \( b \) have no common factors. \( \quad [1] \)

\[ 2 = a^2/b^2 \]

\[ 2b^2 = a^2 \quad [2] \]

Since \( b^2 \in \mathbb{Z} \), we see that \( a^2 \) is even.

By our Lemma, this implies that \( a \) is even.

\[ a = 2k, \text{ some } k \in \mathbb{Z} \]

Substituting into [2]:

\[ 2b^2 = (2k)^2 = 2(2k^2) \]

\[ b^2 = 2k^2 \]

Since \( k^2 \) is an integer, we see that \( b^2 \) is even, hence \( b \) is even by the Lemma.

Since \( a \) and \( b \) are both even, they share a common factor, contradicting [1]. \( \square \).
**Claim:** $(\forall x, y \in \mathbb{N} > 1) [\text{if } x \mid y \text{ then } x \nmid (y + 1)]$

**Proof:**

Let $a, b \in \mathbb{N} > 1$ be selected arbitrarily.

Assume $a \mid b$. [I must show $a \nmid (b + 1).]

Suppose (BWOC) $a \mid (b + 1).

Since $a \mid b$, $(\exists k \in \mathbb{Z}) b = ak$ \hspace{1cm} [1]

Since $a \mid b + 1$, $(\exists m \in \mathbb{Z}) b + 1 = am$

$b = am - 1$

Combining with [1]:

$a k = am - 1$

$1 = am - ak$

$1 = a(m - k)$

[Note that $m > k$, else this contradicts].

$1/a = m - k$. 

Since $m > k$, the R.H.S. is greater than or equal to 1.

Since $a > 1$, the L.H.S. is less than 1. \(\Box\)

Since $a, b$ were selected arbitrarily, the proposition holds for all $x, y \in \mathbb{N} > 1$. \(\Box\)
Claim: There are infinitely many primes.

Proof:
Suppose (BWOC) that there were only finitely many primes. Then there we could enumerate them, like this:
\[ p_1, p_2, p_3, p_4, \ldots p_{k-1}, p_k, \] where \( p_k \) is the last one.
Let \( n = (p_1)(p_2)(p_3)\ldots(p_{k-1})(p_k) + 1 \).
Note that \( n \) is a natural number (since \( \mathbb{N} \) is closed under multiplication and addition.).
By Lemma 3, \( n \) has a prime factor, \( p_i \), for some \( i \leq k \). \[ 1 \]
\[ p_i \mid (p_1)(p_2)(p_3)\ldots(p_{k-1})(p_k), \]
So by Lemma 2, \( p_i \nmid n \).
But this contradicts [1]. \( \boxtimes \) \( \square \)
Claim: The “CodeAnalyzer” program cannot exist.

Proof:
Suppose (BWOC) that there is a “CodeAnalyzer” program.
We will construct another program that relies on CodeAnalyzer:

Program Test(A) { // A is sourcecode for a program
    String result = CodeAnalyzer(A, A)
    If result is ‘‘IT HALTS’’ then do an infinite loop
}

Take the sourcecode for Test and feed it to Test as input.

Case “Test(Test) halts”:
CodeAnalyzer would return “IT HALTS” and so Test(Test) runs forever. ☒

Case “Test(Test) runs forever”: CodeAnalyzer would return “IT RUNS FOR-EVER” and so Test(Test) would terminate. ☒

These cases are exhaustive. □
Claim: All natural numbers are interesting.

Proof:
Suppose (BWOC) that some natural numbers are not interesting. Let \( x \) be the smallest natural number that is not interesting.

\( x \) is the smallest natural number that is not interesting? Wow, that’s interesting!
\( \bigotimes \) \( \square \)