Password for today: beach
Claim: Let \( a, c \in \mathbb{Z}, n \geq 1 \). If \( a \equiv_n c \) then for all \( m \in N \), \( a^m \equiv_n c^m \).

Proof: Let \( a, c, n \) be arbitrarily chosen, and assume \( a \equiv_n c \).
I will induct on \( m \).

Base Case (\( m = 0 \)): \( a^0 = 1 \equiv_n 1 = c^0 \)

Inductive Hypothesis: Assume \( a^k \equiv_n c^k \), for some \( k \in N \).

Inductive Step: [I must show \( a^{k+1} \equiv_n c^{k+1} \) ]

Since we are assuming both \( a \equiv_n c \) and \( a^k \equiv_n c^k \), item #3 in the
Modular Arithmetic Theorem yields:

\[
\begin{align*}
    a(a^k) &\equiv_n c(c^k) \\
    a^{k+1} &\equiv_n c^{k+1}
\end{align*}
\]
Claim: For all $n \in N$, $n^3 \equiv_3 n$.

Proof: I will induct on $n$.

Base Case ($n = 0$): $0^3 = 0 \equiv_n 0$

Inductive Hypothesis: Assume $k^3 \equiv_3 k$, for some $k \in N$.

Inductive Step: [I must show $(k + 1)^3 \equiv_3 k + 1$]

$$(k + 1)^3 = (k + 1)(k^2 + 2k + 1) =$$

$k^3 + 3k^2 + 3k + 1 \equiv_3$

$k^3 + 1 \equiv_3$ [By I.H.]

$k + 1$
Claim: \( (\forall n \geq 1) \left[ \sum_{i=1}^{n} 4i - 2 = 2n^2 \right] \)

Proof: I will induct on \( n \).

Base Case (\( n = 1 \)): \( \sum_{i=1}^{1} [4(i) - 2] = 4(1) - 2 = 2 = 2(1)^2 \)

Inductive Hypothesis: Assume \( \sum_{i=1}^{k} 4i - 2 = 2k^2 \), for some \( k \geq 1 \)

Inductive Step: [I must show \( \sum_{i=1}^{k+1} 4i - 2 = 2(k+1)^2 \)]

\[
\sum_{i=1}^{k+1} 4i - 2 = \left[ \sum_{i=1}^{k} 4i - 2 \right] + 4(k+1) - 2 \quad \text{[By I.H.]} \\
2(k)^2 + 4(k + 1) - 2 = \\
2k^2 + 4k + 2 = \\
2(k + 1)^2 \]
Claim: \((\forall n \geq 1) \left[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right] \)

Proof: I will induct on \(n\).

Base Case \((n = 1)\): \(\sum_{i=1}^{1} i = 1 = \frac{1(2)}{2}\)

Inductive Hypothesis: Assume \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\), for some \(k \geq 1\)

Inductive Step: [I must show \(\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}\)]

\[
\sum_{i=1}^{k+1} i = \left[ \sum_{i=1}^{k} i \right] + (k + 1) = \text{[By I.H.]} \]

\[
\frac{k(k+1)}{2} + k + 1 = \frac{k(k+1) + 2(k+1)}{2} = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}
\]
Claim: \((\forall n \geq 0) \left[ \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \right]\)

Proof: I will induct on \(n\).

Base Case (\(n = 0\)): \(\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2^{0+1} - 1\)

Inductive Hypothesis: Assume \(\sum_{i=0}^{k} 2^i = 2^{k+1} - 1\), for some \(k \geq 0\)

Inductive Step: [I must show \(\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1\)]

\[
\sum_{i=0}^{k+1} 2^i = \left[ \sum_{i=0}^{k} 2^i \right] + 2^{k+1} = [\text{By I.H.}]
\]

\[
2^{k+1} - 1 + 2^{k+1} = 2(2^{k+1}) - 1 = 2^{k+2} - 1
\]
Claim: \((\forall r \in R^{>1})(\forall n \geq 0)\) \[
\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}
\]

Proof: Let \(r\) be chosen arbitrarily. I will induct on \(n\).

Base Case (\(n = 0\)): \[
\sum_{i=0}^{0} r^i = r^0 = 1 = \frac{r^{0+1} - 1}{r - 1}
\]

Inductive Hypothesis: Assume \(\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1}\), for some \(k \geq 0\)

Inductive Step: [I must show \(\sum_{i=0}^{k+1} r^i = \frac{r^{k+2} - 1}{r - 1}\)]

\[
\sum_{i=0}^{k+1} r^i = \left[\sum_{i=0}^{k} r^i\right] + r^{k+1} = [\text{By I.H.}]
\]

\[
\frac{r^{k+1} - 1}{r - 1} + r^{k+1} = \frac{r^{k+1} - 1 + r^{k+1}(r - 1)}{r - 1} = \frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r - 1} =
\]

\[
\frac{r^{k+2} - 1}{r - 1}
\]