Claim: If $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof: Assume $A \subseteq B$. [I must show $\mathcal{P}(A) \subseteq \mathcal{P}(B)$]

Let $X \in \mathcal{P}(A)$. [I must show $X \in \mathcal{P}(B)$]

$X \subseteq A$ (def. of power set)

Since $X \subseteq A$ and $A \subseteq B$, $X \subseteq B$.

$\therefore \ X \in \mathcal{P}(B).$
Claim: If \( n(A) = k \) then \( n(P(A)) = 2^k \).

Proof: I will induct on \( k \).

Base (\( k = 0 \)):
Assume \( n(A) = 0 \).
Then \( A = \emptyset \), so \( P(A) = \emptyset \times \emptyset \), hence \( n(P(A)) = 1 = 2^0 \).

Inductive Hypothesis: Assume (for some \( k \in \mathbb{N} \)) that
\[ n(A) = k \implies n(P(A)) = 2^k. \]

Inductive Step: I must show that for \( k+1 \):
\[ n(A) = k+1 \implies n(P(A)) = 2^{k+1}. \]
Assume \( n(A) = k+1 \). [I must show \( n(P(A)) = 2^{k+1} \)]
Let \( x \in A \); let \( B = A \setminus \{x\} \).

How many subsets of \( A \) do not contain \( x \)?
These are the subsets of \( B \), so \( 2^k \) [using I.H.]

How many subsets of \( A \) contain \( x \)?
These are subsets of \( B \) with \( x \) added, so \( 2^k \).

i. \( n(P(A)) = 2^k + 2^k = 2^{k+1} \).
Lemma: \( x \subseteq A \) and \( x \subseteq B \) \( \iff \) \( x \subseteq A \cap B \). 

Proof: Part I - Assume \( x \subseteq A \) and \( x \subseteq B \). (I will show \( x \subseteq A \cap B \))

Let \( x \in x \). [I must show \( x \in A \cap B \)]

Since \( x \in x \) and \( x \subseteq A \), \( x \in A \)

Since \( x \in x \) and \( x \subseteq B \), \( x \in B \)

Since \( x \in A \) and \( x \in B \), \( x \in A \cap B \)

\( \therefore x \subseteq A \cap B \).

Part II - Assume \( x \subseteq A \cap B \) [Show \( x \subseteq A \) and \( x \subseteq B \)]

Let \( x \in x \).

Since \( x \in x \) and \( x \subseteq A \cap B \), \( x \in A \cap B \)

\( \therefore x \in A \) and \( x \in B \)

\( \therefore x \subseteq A \) and \( x \subseteq B \)
Claim: \( P(A) \cap P(B) = P(A \cap B) \)

Proof:

\[ x \in P(A) \cap P(B) \iff \]
\[ x \in P(A) \text{ and } x \in P(B) \iff \]
\[ x \in A \text{ and } x \in B \iff \]
\[ x \in A \cap B \iff \]
\[ x \in P(A \cap B) \]

\( \_ \_ \_ \)
Selecting a 4-digit PIN:

1) Choose first digit (10 choices)
2) Choose 2nd digit (10 choices)
3) Choose 3rd digit (10 choices)
4) Choose 4th digit (10 choices)

\(10 \times 10 \times 10 \times 10 = 10,000\) ways.

Let \( S = \) all possible PINs. \( n(S) = 10,000 \)
Let \( E = \) "PIN has only even digits."

What is \( n(E) \)?

1) Choose first digit (5 choices)
2) Choose second digit (5 choices)
3) Choose third digit (5 choices)
4) Choose 4th digit (5 choices)

\( n(E) = 5 \times 5 \times 5 \times 5 = 625 \)

\( P(E) = \frac{625}{10000} = 0.0625% \)
Let \( F = " \) all digits in PIN are unique.\( " \)

\[ n(F) : \]
1) Select 1st digit (10 choices)
2) Select 2nd digit (9 choices)
3) Select 3rd digit (8 choices)
4) Select 4th digit (7 choices)

\[ n(F) = \binom{10}{1} \binom{9}{1} \binom{8}{1} \binom{7}{1} = 5040 \]

\[ P(F) = \frac{5040}{10000} = 50.4\% \]

Let \( G = " \) PIN that never has same digit twice in a row.\( " \)

\[ n(G) : \]
1) Select 1st digit (10 ways)
2) Select 2nd digit (9 ways)
3) Select 3rd digit (9 ways)
4) Select 4th digit (9 ways)

\[ n(G) = \binom{10}{1} \binom{9}{1} \binom{9}{1} \binom{9}{1} = 7290 \]

\[ P(G) = 72.9\% \]
S = "all-md license plates"

\[ n(S) = 10^5 \cdot 26^2 = 67,600,000 \]

E = "plate with matching digits and letters"

\[ n(E) = 260 \]

\[ P(E) = \frac{260}{67,600,000} = 0.00000384615 \]

or 1 in 260,000