DIMACS Series in Discrete Mathematics
and Theoretical Computer Science
Volume 6, 1991

# The Densest Double-Lattice Packing of a Convex Polygon

DAVID M. MOUNT

ABSTRACT. A lattice packing of a planar body is an infinite packing of the plane by translated copies of the body where the copies are translated to the points of a lattice. A double-lattice packing is a union of two lattice packings such that a $180^\circ$ rotation about some point interchanges the two packings. We show that the densest double-lattice packing of an $n$-sided convex polygon can be computed in $O(n)$ time.

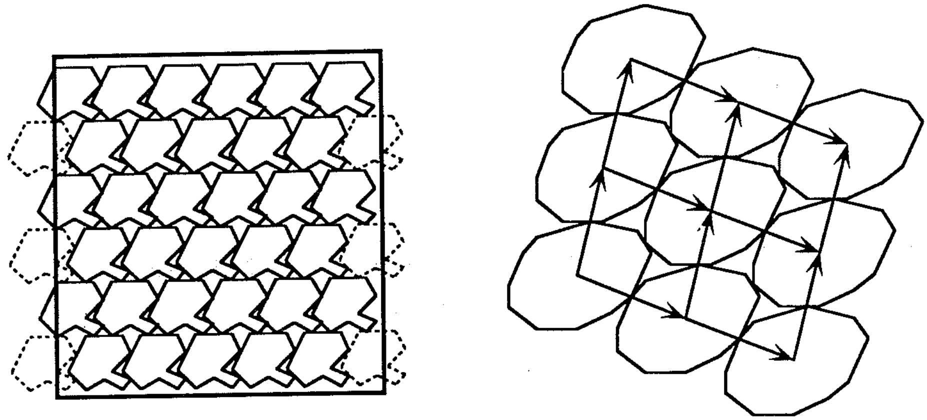
## 1. Introduction

Packing problems such as the knapsack problem and bin packing problem are well known in the fields of algorithm design and operations research because of their many applications to problems such as stock-cutting in computer-aided manufacturing. Because general formulations of these problems are known to be NP-complete [5], it is of interest to discover formulations of these problems which are solvable in polynomial time and yet are of general enough interest to be useful in applications. One such formulation is that of finding the densest (infinite) packing of congruent copies of a single polygon in the plane. More formally, given a simple polygon $P$, the problem is to determine an infinite collection of rigidly transformed copies of $P$ in the plane having pairwise disjoint interiors, such that *density* of the system (intuitively the fraction of the plane covered by copies) is maximized. This problem is of interest in packing applications where a large number of identical 2-dimensional objects are to be packed into a large container. If the size of the objects is small relative to the size of the container, then a reasonable heuristic is to determine the densest infinite packing of the objects in the plane, and then truncate the packing to fit the container (see Figure 1(a)).

---

1991 *Mathematics Subject Classification.* Primary 68U05; Secondary 68Q25.
This research has been partially supported by National Science Foundation grant CCR–89–08901.

©1991 American Mathematical Society
1052-1798/91 $1.00 + $.25 per page



(a) (b)

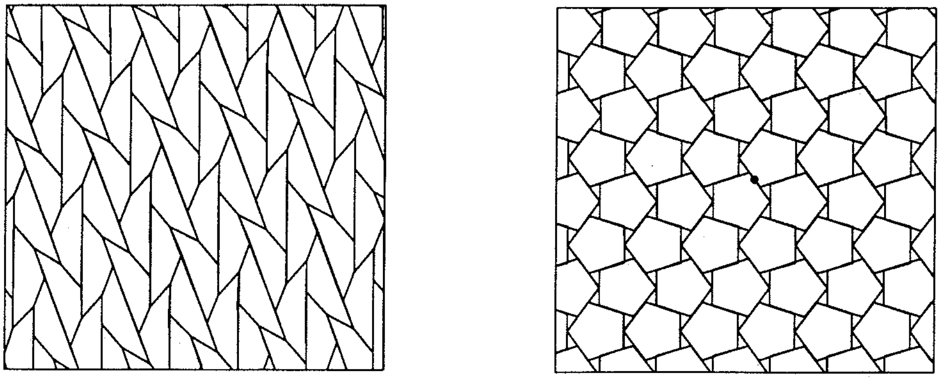
FIGURE 1. Lattice packings.

Mount and Silverman showed that for an $n$-sided convex polygon, the densest packing in the plane, allowing only translations not rotations, could be computed in $O(n)$ time. In proving this result they applied a classical theorem due to Rogers **[14]** which states that the densest packing by translates of a convex body is generated by a *lattice*, that is, a system of points defined by all integer linear combinations of two independent vectors (see Figure 1(b)). They showed how to compute the densest lattice packing in $O(n)$ time.

One major shortcoming in Mount and Silverman's result is that it does not allow objects to be rotated. Although this is reasonable for packing applications where the packing domain has a directional grain (e.g. when cutting fabric), better packing densities are achievable if rotation is allowed. For example, if the objects to be packed are triangles then the densest packing by translates has density $2/3$ **[2]**, while if rotation is allowed then a packing of density 1, a *tiling*, is possible by mating each triangle with a $180^\circ$ rotation of itself to form a parallelogram, and then tiling the plane with these parallelograms.

When rotation is allowed we know of no general simple structure, such as the lattice, which is guaranteed to generate the densest packing. For example, Figure 2(a) shows an example due to Heesch of a pentagon which, if allowed to rotate, can tile the plane with a periodic structure [7] (see also **[6, p. 31]**). Perhaps the simplest packing structure which allows rotation is a *double-lattice* packing, which is the union of two lattice packings, such that a rotation by $180^\circ$ about some point interchanges the two packings. The packing of Figure 2(b) is an example of a double-lattice packing of a regular pentagon, and the triangle tiling described earlier is also an example.

In this paper we describe an $O(n)$ algorithm for determining the densest



(a) (b)

FIGURE 2. Packing with rotation.

double-lattice packing of a convex $n$-sided polygon $P$. Our result is based on a reduction due to G. Kuperberg and W. Kuperberg [**9**]. They showed that the densest double-lattice packing of a convex body can be derived by finding a certain type of inscribed parallelogram, called an *extensive parallelogram*, of minimum area.

The problem of finding the densest packings of a convex object in Euclidean space has a rich history. See Rogers [**15**] and Fejes-Toth [**3**] for surveys of this field. The related subject of tilings has been also been studied [**6**]. There have been relatively few computational results in this area. In addition to Mount and Silverman's result, De Pano described a linear time algorithm for packing congruent copies of a convex polygon in the plane (where rotations are allowed) such that the density of the resulting packing is at least 3/4 [**1**]. The packing generated by De Pano's algorithm (which is based on a construction given by Kuperberg [**10**]) is also a double-lattice packing, but it is not necessarily the densest double-lattice packing for the given input.

In §2 we present the geometrical underpinnings of the algorithm and show how to reduce the packing problem to a problem of finding a certain minimum area inscribed parallelogram, called the *half-length* parallelogram. In §3 we derive a rotating calipers algorithm for finding this parallelogram and analyze the algorithm's running time.

## 2. Double-lattice packings and parallelograms

Throughout this paper $P$ will denote an $n$-sided convex polygon in the real plane, $\mathbf{R}^2$. For $v \in \mathbf{R}^2$, the *translate* of $P$ by $v$, $P+v$, is the set of points $\{p+v | p \in P\}$. Let $-P$ denote the set $\{-x | x \in P\}$, a rotation of $P$ through $180°$ about the origin. For a given pair of linearly independent vectors $u$ and $v$, the *lattice* generated by $u$ and $v$ is the set of vectors

$$L(u, v) = \{iu + jv \mid i \text{ and } j \text{ integers}\}.$$

The vectors $u$ and $v$ span a *basic parallelogram* of the lattice.

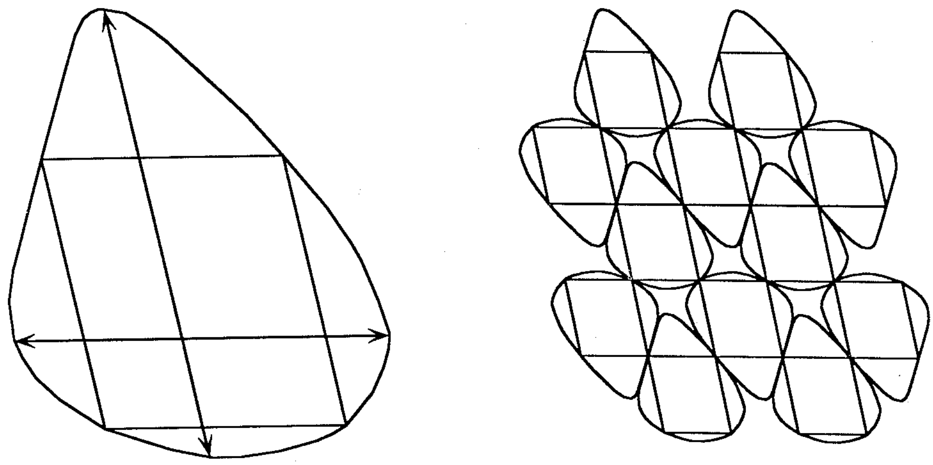
Consider an infinite system of bodies resulting by translating $P$ by each vector in a lattice, $L(u, v)$. If the interiors of this system are pairwise disjoint, then we say that $u$ and $v$ define a *lattice packing* of $P$. The *density* of a lattice packing is the ratio of the area of $P$ to the area of the basic parallelogram, the absolute value of $\det(u, v)$. The density of a packing is at most 1, where equality occurs if the packing is a tiling of the plane (implying that $P$ is either a parallelogram or a centrally symmetric hexagon).

A *double-lattice packing* is the union of two lattice packings such that a $180°$ rotation about some point interchanges these two packings. (Figure 2(b) gives an example. A possible point of rotation is shown.) It is easy to see that the density of a double-lattice packing is the ratio of twice the area of $P$ to the area of the basic parallelogram.

G. Kuperberg and W. Kuperberg **[9]** showed that there exists a double-lattice packing for any convex body $P$ of density at least $\sqrt{3}/2$, matching existing lower bound for lattice packings of centrally symmetric convex bodies **[11, 4]**. For convex polygons with three or four sides, there exist double-lattice packings that tile the plane. The Kuperberg's also conjecture that the densest packings by congruent copies of regular pentagons and regular heptagons are double-lattice packings. It is not hard to show that any lattice packing of a convex body can be converted into a double-lattice packing of equal density, and hence double-lattice packings can achieve at least as good densities as lattice packings for any given convex polygon.

Given a convex polygon $P$, a *chord* of $P$ is any line segment whose endpoints lie on the boundary of $P$. Define the *angle* of a line segment of nonzero length to be the arctangent of the slope of the line segment normalized to the interval $[0°, 180°)$. Given an angle $\theta$, the *length* of $P$ at angle $\theta$ is the length of the longest chord of $P$ whose angle is $\theta$, and the *width* of $P$ at the angle $\theta$ is the perpendicular distance between the two parallel lines of support for $P$ whose angle is $\theta$. A chord is of maximal length for a given angle if and only if there exist two parallel lines of support for $P$ that pass through the endpoints of the chord. We call such a chord a *$\theta$-diameter*. The diameter chord at a given angle need not be unique, but it is easy to see that for any angle $\theta$ there exists a $\theta$-diameter such that at least one endpoint of the chord coincides with a vertex of $P$.

An inscribed parallelogram is said to be *extensive* if the length of each of its sides is at least one-half the length of the diameter in the same direction as that side (see Figure 3(a)). Kuperberg and Kuperberg showed that there is a close relationship between dense double-lattice packings and extensive parallelograms **[9]**. They observed that if $Q$ is an extensive parallelogram inscribed in $P$, then a double-lattice packing for $P$ can be generated as follows. Translate $Q$ and $P$ simultaneously so that one of the vertices of $Q$ coincides with the origin, and let $u$ and $v$ be the vectors that span $Q$. By



(a) (b)

FIGURE 3. Extensive parallelogram and double-lattice packing.

translating $P \cup -P$ to each point of the lattice generated by $2u$ and $2v$, it follows from the convexity of $P$ and the definition of extensive parallelogram that the resulting system is a double-lattice packing for $P$ (see Figure 3(b)).

THEOREM (G. Kuperberg and W. Kuperberg). *If $P$ is a convex body, there exists a densest double-lattice packing for $P$ which is generated by a minimum area extensive parallelogram inscribed in $P$.*

Two distinct parallel chords of equal length in $P$ define a parallelogram inscribed within $P$. If the angle of these chords is $\theta$ then we call this parallelogram a $\theta$-parallelogram. The parallel chords are the *bases* of the $\theta$-parallelogram. The *length* of a $\theta$-parallelogram is the length of its bases, and the *width* of a $\theta$-parallelogram is the perpendicular distance between the two lines containing its bases. The area of a $\theta$-parallelogram is just the product of its length and width. A *half-length* $\theta$-parallelogram for $P$ is a $\theta$-parallelogram whose bases are half the length of the $\theta$-diameter of $P$ (see Figure 4).

A convex planar body is said to be *strictly convex* if its boundary contains no line segments. Observe that for a strictly convex body (i.e., a convex body containing no line segment on its boundary) there are exactly two chords parallel to and of half the length of the $\theta$-diameter, and thus there is a unique half-length $\theta$-parallelogram. If a convex polygon has one or two edges which are parallel to and greater than half the length of the $\theta$-diameter then there may be infinitely many half-length $\theta$-parallelograms. These parallelograms arise by selecting the bases of the parallelogram to be any subsegment of the

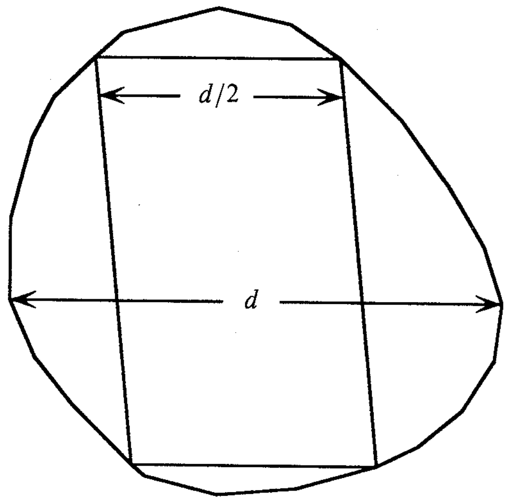



FIGURE 4. The half-length parallelogram.

appropriate length from one of these parallel edges (an example is shown in Figure 8 on p. 256). Since all parallelograms generated in this manner have equal area, for our purposes it suffices to select any one of them arbitrarily, e.g. by sliding its bases as far counterclockwise along this edge as possible. By assuming that all $\theta$-parallelograms are slid into such a canonical configuration, we can talk about the *unique* half-length $\theta$-parallelogram for a given $\theta$.

The Kuperberg's remark without proof **[9]** that in order to compute the minimum area extensive parallelogram inscribed in $P$, it suffices to consider the set of half-length $\theta$-parallelograms for each $\theta$ between 0 and $180°$. (By symmetry the $\theta$-parallelograms repeat cyclically with period $180°$.) For the sake of completeness we present a proof of this remark.

THEOREM 2.1. *A minimum area extensive parallelogram inscribed in a convex polygon $P$ is achieved by a half-length $\theta$-parallelogram for some angle $\theta$ between 0 and* $180°$.

PROOF. It suffices to prove that for every $\theta$, (1) every half-length $\theta$-parallelogram is extensive, and (2) an extensive $\theta$-parallelogram is locally minimal if and only if it is a half-length $\theta$-parallelogram.

To show (1) let $a$ and $b$ be the bases of a half-length $\theta$-parallelogram $Q$, and let $c$ be a $\theta$-diameter. Consider a convex quadrilateral $R$ which is bounded by the two lines passing through the left endpoint of $c$ and the left endpoints of $a$ and $b$, and the two lines passing through the right endpoint of $c$ and the right endpoints of $a$ and $b$ (see Figure 5(a)). Because $a$ and $b$ are parallel to and of half the length of $c$, it follows that the endpoints of $a$ and $b$ lie on the midpoints of the sides of $R$. Thus $Q$ is extensive for $R$ since its width and length are exactly one-half the width and length of $R$,

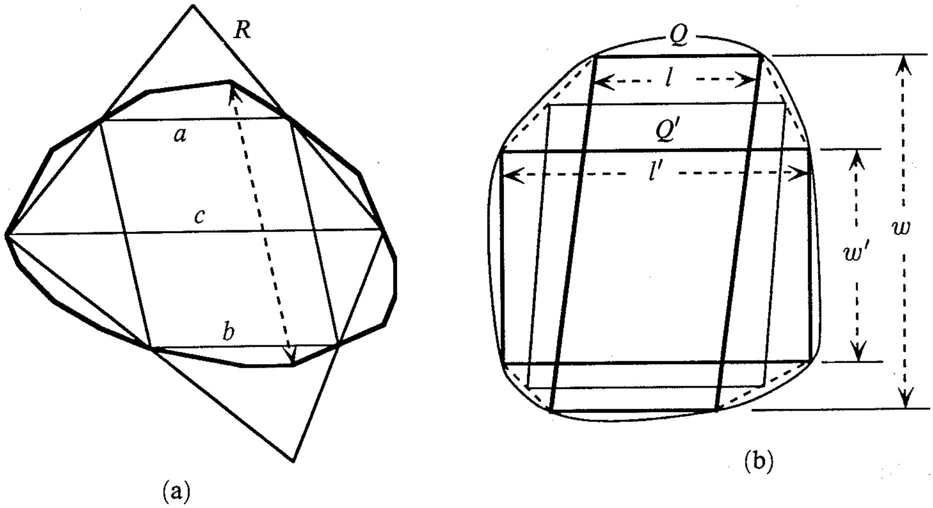



FIGURE 5. Minimality of the half-length parallelogram.

respectively. Consider the angle $\phi$ of the sides of $Q$ other than $a$ and $b$. It is a simple consequence of convexity that the longest chord for $P$ at the angle $\phi$ lies entirely within $R$. It follows that the width of $Q$ is at least half the width of $P$. Since $P$ and $R$ have the same $\theta$-diameter, the length of $Q$ is at least half the length of $P$. Therefore $Q$ is extensive for $P$.

We will prove (2) for strictly convex bodies. (Although $P$ is not strictly convex, it can be approximated by a convergent sequence of strictly convex bodies, and we can select a convergent subsequence of minimum extensive parallelograms.) If $P$ is strictly convex, then there is a unique $\theta$-parallelogram for each given length. As the length increases, the width of the $\theta$-parallelograms decrease. We show that the area of the $\theta$-parallelograms, as a function of length, is upward convex. Hence the minimum of this function over any interval is achieved at an endpoint of the interval. Thus the minimum extensive parallelogram must be a half-length $\theta$-parallelogram.

Consider two lengths $l$ and $l'$, $0 < l < l' < |c|$. Let $Q$ and $Q'$ be the $\theta$-parallelograms with these respective lengths. Let $w$ and $w'$ be the respective widths of $Q$ and $Q'$ (see Figure 5(b)). Consider the four lines segments which join each of the four vertices of $Q$ to its corresponding vertex of $Q'$. For each $p$, $0 \le p \le 1$, let $q = 1 - p$. There is an interpolated parallelogram of length $pl + ql'$ and width $pw + qw'$ whose vertices lie on these four lines segments. (The endpoints of this parallelogram are weighted averages of corresponding pairs of endpoints of $Q$ and $Q'$.)

By the convexity of $P$, each interpolated parallelogram is enclosed within $P$, and so the $\theta$-parallelogram of the same length, $pl + ql'$, has area no smaller than this interpolated parallelogram. Hence it suffices to show that the area of the interpolated parallelogram is not less than the corresponding weighted average of the areas of $Q$ and $Q'$, that is

$$p(lw) + q(l'w') \leq (pl + ql')(pw + qw').$$

To prove this, first observe that because $l < l'$ and $P$ is strictly convex we have $w > w'$. Clearly $0 \leq pq \leq 1$. Thus we have

$$0 \leq pq(l' - l)(w - w').$$

By simple manipulations and the facts that $1 - p = q$ and $1 - q = p$ we get

$$\begin{aligned} pq(l' - l)(w - w') &= (pl + ql')(pw + qw') - (plw + ql'w') \\ 0 &\leq (pl + ql')(pw + qw') - (plw + ql'w') \\ p(lw) + q(l'w') &\leq (pl + ql')(pw + qw'), \end{aligned}$$

completing the proof. □

## 3. The algorithm

In the previous section we introduced the notion of a half-length parallelogram inscribed in the $n$-sided convex polygon $P$ and showed that the problem of finding the densest double-packing can be reduced to computing the minimum area half-length parallelogram. In this section we show how to compute this minimum area half-length parallelogram. We employ the technique of *rotating calipers* **[16]**. For each angle $\theta$, $0^\circ \leq \theta < 180^\circ$, we compute (explicitly or implicitly) a representative half-length $\theta$-parallelogram (recalling our assumption that we can break ties among multiple half-length $\theta$-parallelograms arbitrarily since they have equal area). It suffices to consider only 180 degrees of rotation because the half-length parallelogram in the directions $\theta$ and $180^\circ + \theta$ are equal.

As in all rotating caliper algorithms, we define a finite set of critical angles $\theta$ at which we explicitly compute the half-length $\theta$-parallelogram. An angle $\theta$ is said to be *critical* if either (1) both endpoints of the diameter chord in the direction $\theta$ coincide with vertices of the polygon (recall that we may always assume that at least one endpoint of the diameter chord coincides with a vertex of $P$), or (2) an endpoint of the $\theta$-parallelogram coincides with a vertex of $P$. We will show that between any two consecutive critical angles, the diameter and the half-length parallelograms vary in a simple continuous way. This fact will allow us to compute the next critical angle in $O(1)$ time, and to determine the minimum area half-length parallelogram between a pair of consecutive critical angles in $O(1)$ time. In addition we will show that as $\theta$ increases, the points of contact between the half-length $\theta$-parallelogram and the diameter in the direction $\theta$ will move monotonically counterclockwise around the boundary of $P$. From this property, which is called the *interspersing property*, it will follow that there are $O(n)$ critical angles.

The algorithm consists of three basic steps, where the second and third steps are repeated until $\theta \geq 180^\circ$.

**Initialization:** Compute an initial half-length $\theta$-parallelogram and a diameter for $\theta = 0^\circ$.

**Advancing to the next critical angle:** Given an arbitrary $\theta$, a half-length $\theta$-parallelogram, and a $\theta$-diameter, determine the next critical angle $\theta' > \theta$ and determine the half-length $\theta'$-parallelogram and the $\theta'$-diameter.

**Minimizing between critical angles:** Between a pair of consecutive critical angles $\theta$ and $\theta'$ determine the minimum area half-length parallelogram for all angles in the interval $[\theta, \theta']$.

Each of these steps will be treated separately in this section.

**3.1. Initialization.** Let us first consider the problem of finding the initial half-length $\theta$-parallelogram and $\theta$-diameter for $\theta = 0$. Consider a topmost vertex of $P$ (with maximum $y$-coordinate) and a bottommost vertex of $P$ (with minimum $y$-coordinate). These two vertices subdivide the boundary of $P$ into a left side and a right side. By shooting a horizontal ray from each vertex on one side to the opposite side, we decompose the interior of $P$ into a sequence of trapezoids with horizontal bases (where the topmost and bottommost trapezoids may degenerate to triangles) (see Figure 6). This trapezoidal decomposition can be computed in $O(n)$ time by merging the sorted lists of vertices on the left side of $P$ with the vertices of the right side by $y$-coordinate.

By scanning through this list of trapezoids, it is an easy matter to find the maximum trapezoid base, which forms the horizontal diameter of $P$, and to find the two trapezoids, one lying above and one lying below the diameter, which contain the horizontal chords whose length is one half the length of the diameter. These chords are the bases for the initial half-length parallelogram.

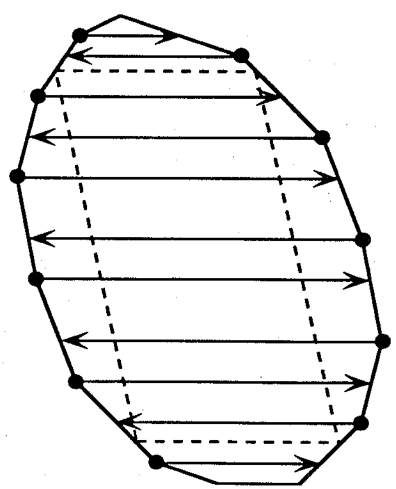


FIGURE 6. Computing the initial half-length parallelogram.

The total running time of this algorithm is clearly $O(n)$. (Observe that because the horizontal width function is convex, we could have computed the initial half-length parallelogram in $O(\log^2 n)$ time by performing a type of binary search on the $y$-coordinates of the polygon. However, this linear time algorithm is simpler and suffices for our purposes.)

**3.2. Advancing to the next critical angle.** Next we consider how vertices of the half-length parallelogram move incrementally as the directional angle $\theta$ rotates through a small angle starting with some initial value $\theta_0$. We use the notation $\overrightarrow{ab}$ to denote the directed segment from point $a$ to point $b$. By translating the tail of the segment to the origin, a directed segment $\overrightarrow{ab}$ can naturally be identified with the vector $b - a$. We will think of the edges of $P$ as segments directed counterclockwise around the boundary of $P$, so that the *tail* and *head* of an edge are the clockwise and counterclockwise vertices of the edge, respectively. We consider each edge of $P$ to be closed at its tail and open at its head, so that a vertex belongs to the edge following it in counterclockwise order about the boundary.

We begin by analyzing the movement of the diameter chord as $\theta$ increases. This analysis was given by Toussaint in describing his rotating calipers algorithm for finding the diameter of a convex polygon **[16]**, but we repeat it here for completeness. Let $c_1(\theta)$ and $c_2(\theta)$ denote the endpoints of a $\theta$-diameter (see Figure 7). Recall that there exist two parallel lines of support for $P$ passing through the endpoints of this chord. If either endpoint lies in the interior of some edge, then these support lines are uniquely determined. If both of the endpoints lie on vertices of $P$, then there may be an infinite number of parallel support lines to choose from. We make the convention of selecting the extreme counterclockwise angle for these lines, and thus we are assured that at least one of the lines of support will be colinear with one of the edges following $c_1(\theta)$ or $c_2(\theta)$ in counterclockwise order. This selection can be made in constant time by analyzing the angles of the edges of $P$ which lie clockwise of the current diameter chord's endpoints.

Suppose that the one of the support lines is colinear with the edge $e$ of $P$ which lies just counterclockwise of $c_1(\theta)$. As $\theta$ increases, the point $c_2(\theta)$ remains fixed and $c_1(\theta)$ travels counterclockwise along the edge $e$ (see Figure 7(a)). The supporting lines for the diameter do not change until $c_1(\theta)$ reaches the vertex at the head of $e$. If this is not the case, then the other support line is colinear with the edge which lies just counterclockwise of $c_2$. In this case $c_1(\theta)$ remains fixed and $c_2(\theta)$ travels monotonically along this edge until reaching the next vertex (see Figure 7(b)). The angle at which this event occurs is denoted $\theta_c$. Clearly $\theta_c > \theta_0$.

Next we consider how the endpoints of the bases of the $\theta$-parallelogram vary with $\theta$. (As we will see later, we do not deal with $\theta$ directly as an angle measured, say, in degrees. Rather the varying angle will be expressed parametrically and all quantities depending on angles will be simple linear

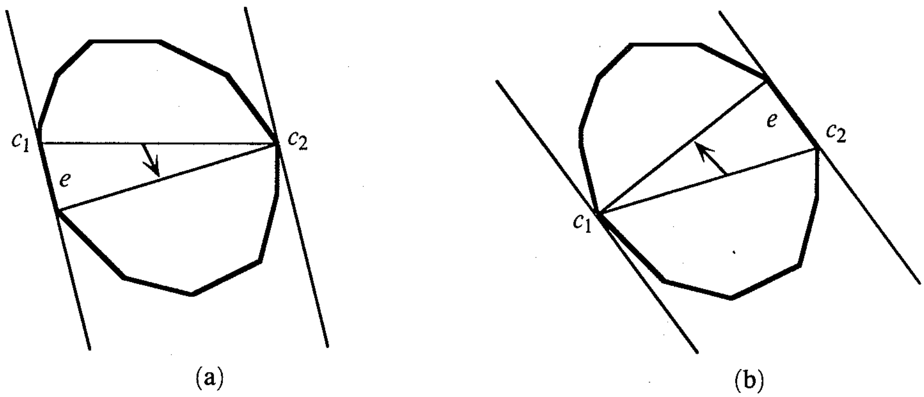



FIGURE 7. The movement of the diameter chord.

functions.) We first define two intermediate values that will be helpful. For $\theta_0 \le \theta \le \theta_c$, $C(\theta)$ denotes the vector which is one-half the length of the diameter chord at the angle $\theta$, and let $D(\theta)$ be the net motion of head of this vector as $\theta$ increases from $\theta_0$. That is,

$$C(\theta) = \frac{1}{2}(c_2(\theta) - c_1(\theta)), \qquad D(\theta) = C(\theta) - C(\theta_0).$$

For $\theta$ in the range $\theta_0 \le \theta \le \theta_c$, the quantities $c_1(\theta)$, $c_2(\theta)$, $C(\theta)$ and $D(\theta)$ can be computed in constant time. Clearly $D(\theta_0) = 0$. If $c_1'$ and $c_2'$ denote the heads of the edges on which $c_1$ and $c_2$ lie, then $D(\theta_c)$ is either equal to $(1/2)(c_1' - c_1)$ or $(1/2)(c_2 - c_2')$, depending on whether the $c_1$ or $c_2$ is the moving endpoint of the diameter chord.

Let $a_1$ and $a_2$ denote the the endpoints of one of the base chords of the half-length $\theta_0$-parallelogram. (An analogous construction can be applied to the other base chord from $b_1$ to $b_2$.) Let $a_1'$ and $a_2'$ denote the heads of the edges on which $a_1$ and $a_2$ lie. For the sake of illustration, assume that the chords $\overrightarrow{a_1a_2}$ and $\overrightarrow{b_1b_2}$ are horizontal and directed from left to right. Let $a_1'$ and $a_2'$ denote the heads of the edges on which $a_1$ and $a_2$ lie, respectively. For $i = 1, 2$, let $h_i$ denote the directed segment $\overrightarrow{a_ia_i'}$. Notice that $h_i$ is of nonzero length (by our convention that a vertex of $P$ lies on the next edge in counterclockwise order) and $h_i$ is directed counterclockwise about the boundary $P$.

For an angle $\theta$, let $a_i(\theta)$, denote the position of the corresponding endpoint for the half-length $\theta$-parallelogram. We will show that as $\theta$ increases from $\theta_0$, $a_i(\theta)$ travels monotonically along $h_i$, and furthermore the length of the motion vector $a_i(\theta) - a_i(\theta_0)$ is related to the length of $D(\theta)$ by a constant scale factor.

Suppose that the segments $h_1$ and $h_2$ are parallel to one another. Observe that because $P$ is convex and $\overrightarrow{a_1a_2}$ is parallel to and of strictly lesser length

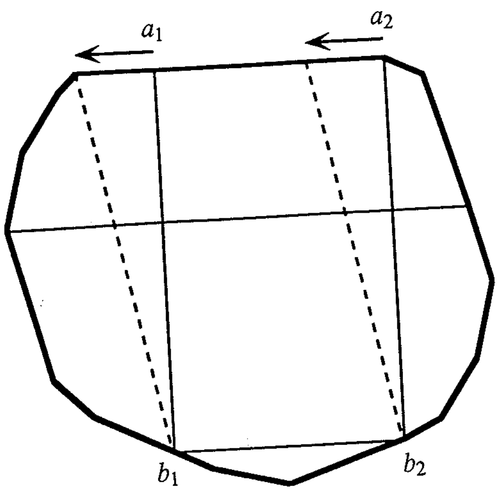



FIGURE 8. Sliding along an edge of $P$.

than the diameter, $a_1$ and $a_2$ must lie on the same edge of $P$ (see Figure 8). This edge is parallel to the diameter chord. If this is the case, as mentioned earlier, we map this $\theta_0$-parallelogram into canonical position by "sliding" the chord $\overrightarrow{a_1 a_2}$ counterclockwise along this edge until the left endpoint $a_1$ is coincident with a vertex of $P$. (For the case of the chord $\overrightarrow{b_1 b_2}$ it will be the right endpoint $b_2$ which becomes coincident with a vertex of $P$.) This transformation does not alter the area of the parallelogram. After sliding, we reevaluate $h_1$ and $h_2$. By our convention that a vertex belongs to the next counterclockwise edge, it follows that $h_1$ and $h_2$ are no longer parallel.

Let $H_1$ and $H_2$ denote the vectors corresponding to the directed segments $h_1$ and $h_2$. The parameters $\alpha_1$ and $\alpha_2$ which will define the incremental motion of $a_1(\theta)$ and $a_2(\theta)$ are introduced in the following lemma.

LEMMA 3.1. *There exist two unique nonnegative constants $\alpha_1$ and $\alpha_2$, at least one of which is nonzero, such that $D(\theta_c) = \alpha_2 H_2 - \alpha_1 H_1$.*

PROOF. Since $h_1$ and $h_2$ are not parallel to each other and are of nonzero length, $H_1$ and $H_2$ form a basis for $R^2$. Because $D(\theta_c)$ is nonzero there exist two unique $\alpha_1$ and $\alpha_2$ which cannot both be equal to zero such that $D(\theta_c) = \alpha_2 H_2 - \alpha_1 H_1$. To see that both $\alpha_1$ and $\alpha_2$ are nonnegative, imagine for the sake of concreteness that the diameter chord is horizontal and directed from left to right and that $\overrightarrow{a_1 a_2}$ lies horizontally above this chord. Clearly $a_1$ lies on the left side of $P$'s boundary and $a_2$ lies on the right side of $P$'s boundary, and thus the vectors $-H_1$ and $H_2$ must each have nonnegative vertical components (see Figure 9). Because $D(\theta_c)$ is parallel to the supporting lines passing through the endpoints of the diameter chord, it follows from convexity that $D(\theta_c)$ lies within the minor angle subtended by $-H_1$ and $H_2$. Thus $\alpha_1$ and $\alpha_2$ are both nonnegative. □

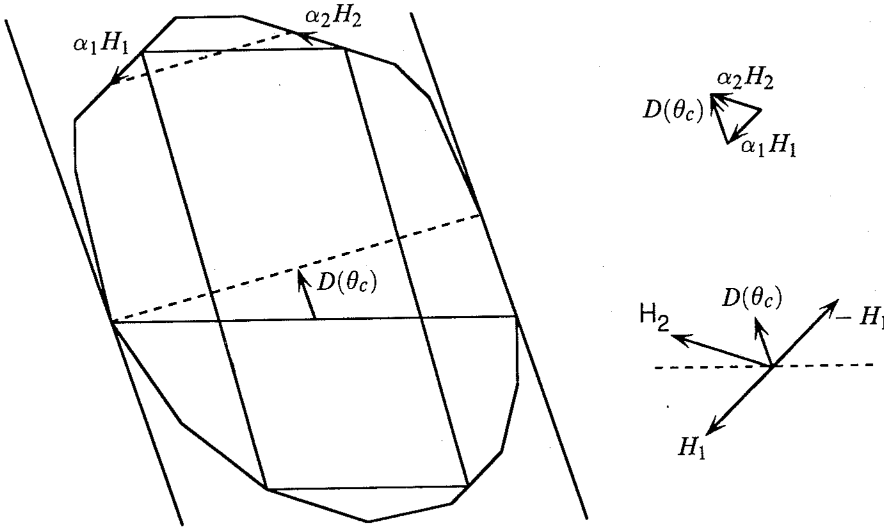



FIGURE 9. Rotating a base chord of the half-length parallelogram.

Rather than deal with angles and trigonometric functions directly, we can instead associate each angle $\theta$ in the range $\theta_0 \leq \theta \leq \theta_c$ uniquely with a parameter $0 \leq t \leq 1$. In particular, we can associate $\theta$ with the ratio $t(\theta)$ of the lengths of the (parallel) vectors $D(\theta_c)$ to $D(\theta)$. In other words, $t(\theta)$ is uniquely defined by the equation $t(\theta)D(\theta_c) = D(\theta)$. Observe that since $D(\theta_c)$ is nonzero, $t(\theta)$ is a monotonically strictly increasing function of $\theta$.

Using this parametric representation of angles, for all sufficiently small angles (essentially up to the next critical angle), we can define the continuous motion of the endpoints of each base of the half-length $\theta$-parallelogram as a simple linear function of this parameter.

LEMMA 3.2. *Let*

$$t_a = \frac{1}{\max(\alpha_1, \alpha_2)}.$$

*For each angle $\theta$ for which $0 \leq t(\theta) \leq \min(t_a, 1)$, we have the following:*

(i) *The endpoints $a_1(\theta)$ and $a_2(\theta)$ lie on the respective segments $h_1$ and $h_2$.*

(ii) *As $\theta$ increases these endpoints move monotonically counterclockwise along these segments, and in particular*

$$a_1(\theta) = a_1 + t(\theta)\alpha_1 H_1, \qquad a_2(\theta) = a_2 + t(\theta)\alpha_2 H_2.$$

(iii) *For the value of $\theta$ for which $t(\theta) = t_a$, at least one of the endpoints of the base $\overrightarrow{a_1(\theta)a_2(\theta)}$ coincides with a vertex of $P$.*

PROOF. First observe that because $\alpha_1$ and $\alpha_2$ are both nonnegative and both cannot be zero, $t_a$ is defined and positive. Define $\theta_a$ to be the angle for which $t(\theta_a) = t_a$. Consider any $\theta$, $\theta_0 \le \theta \le \min(\theta_a, \theta_c)$. For these values of $\theta$, we have $0 \le t(\theta) \le \min(1, t_a)$. For $i = 1, 2$, by simple substitution we have $a_i(\theta) = a_i + t(\theta)\alpha_i H_i$. Clearly $t(\theta)\alpha_i \le 1$ so $a_i(\theta)$ lies somewhere along the segment $h_i$, establishing (i). Indeed, if $\theta = \theta_a$, then either $a_1(\theta)$ or $a_2(\theta)$ coincides with the head of its respective edge (depending on the $\alpha$-value which dominates in the definition of $t_a$), establishing (iii).

To show that $a_2(\theta)$ and $a_1(\theta)$ are the endpoints of the base chord of the half-length $\theta$-parallelogram, it suffices to show that $a_2(\theta) - a_1(\theta) = C(\theta)$.

$$\begin{aligned} a_2(\theta) - a_1(\theta) &= (a_2 + t(\theta)\alpha_2 H_2) - (a_1 + t(\theta)\alpha_1 H_1) \\ &= (a_2 - a_1) + t(\theta)(\alpha_2 H_2 - \alpha_1 H_1) \\ &= C(\theta_0) + t(\theta)D(\theta_c) \\ &= C(\theta_0) + D(\theta) \\ &= C(\theta). \end{aligned}$$

Together with the observation that this motion function is positive and linear in $t$ establishes (ii). □

By applying a similar construction for the base chord $\overrightarrow{b_1 b_2}$ we derive an angle $\theta_b > \theta_0$ at which the moving chord $\overrightarrow{b_1(\theta) b_2(\theta)}$ first encounters an endpoint of $P$. Let

$$\theta^* = \min(\theta_a, \theta_b, \theta_c).$$

For all angles $\theta$, $\theta_0 \le \theta < \theta^*$, each endpoint of the $\theta$-diameter and each vertex of the half-length $\theta$-parallelograms moves along a single edge of $P$. Furthermore, the exact location of these endpoints and vertices can be determined by simple linear combinations as shown in Lemma 3.2. At the angle $\theta^*$ at least one of these points coincides with a vertex of $P$, hence $\theta^*$ is the next critical angle. The vertex which becomes critical depends on which of $\theta_a$, $\theta_b$ and $\theta_c$ is minimum. At this point the edges of contact with the vertices must be reevaluated, and we are ready to repeat the process to find the next critical angle.

We can now summarize the algorithm for advancing from the half-length parallelogram at some angle $\theta_0$ (not necessarily a critical angle) to the next critical angle. Let $\overrightarrow{c_1 c_2}$ denote the current diameter chord and let $\overrightarrow{a_1 a_2}$ and $\overrightarrow{b_1 b_2}$ denote the current base chords for the half-length parallelogram. We observe that at no time is it necessary to compute the actual angles at which the given events occur, but rather to compute the parameters $t_a$ and $t_b$ from which these angles are derived. The parameter value $t = 1$ corresponds to the motion which rotates the diameter chord to its next critical placement.

ALGORITHM. (Advancing to the next critical half-length parallelogram):

(1) Rotate the lines of support for $P$ passing through $c_1$ and $c_2$ to their most counterclockwise orientation. If one of the support lines is colinear with the edge on which $c_1$ lies, then let $D := (1/2)(c_1 - c_1')$,

where $c_1'$ is the head of the edge on which $c_1$ lies. Otherwise let $D := (1/2)(c_2' - c_2)$ where $c_2'$ is the head of the edge on which $c_2$ lies. In the first case $c_1$ is the moving endpoint of the diameter, and in the other case $c_2$ is the moving endpoint. ($D$ is $D(\theta_c)$ defined earlier.)

(2) If $a_1$ and $a_2$ both lie on the same edge of $P$, then slide these endpoints counterclockwise along this edge while maintaining a constant distance between the two of them until one reaches a vertex of $P$. Determine the new edge on which this vertex lies.

(3) Let $a_1'$ and $a_2'$ denote the heads of the edges on which $a_1$ and $a_2$ lie, respectively. Let $H_1 := a_1' - a_1$, and let $H_2 := a_2' - a_2$.

(4) Determine constants $\alpha_1$ and $\alpha_2$ such that $D = \alpha_2 H_2 - \alpha_1 H_1$. (These constants describe the relative movement of $a_1(\theta)$ and $a_2(\theta)$ along the segments $h_1$ and $h_2$ in terms of the movement of the diameter endpoint.)

(5) Let $t_a := 1/\max(\alpha_1, \alpha_2)$. (This is the maximum motion parameter before $a_1(\theta)$ or $a_2(\theta)$ reaches a vertex of $P$.)

(6) Repeat steps (2) through (5) replacing $b_1$ and $b_2$ for $a_1$ and $a_2$, respectively. Let $\beta_1$, $\beta_2$, and $t_b$ denote the results.

(7) Let $t^* := \min(t_a, t_b, 1)$. This determines the motion to the next critical event.

(8) Let

$$\begin{aligned} a_1 &:= (1-\alpha_1 t^*)a_1 + \alpha_1 t^* a_1' & \quad a_2 &:= (1-\alpha_2 t^*)a_2 + \alpha_2 t^* a_2' \\ b_1 &:= (1-\beta_1 t^*)b_1 + \beta_1 t^* b_1' & \quad b_2 &:= (1-\beta_2 t^*)b_2 + \beta_2 t^* b_2'. \end{aligned}$$

If $c_1$ is the moving endpoint of the diameter chord then let $c_1 := (1-t^*)c_1 + t^* c_1'$ and otherwise let $c_2 := (1-t^*)c_2 + t^* c_2'$.

A *convex combination* of two points (vectors) $a$ and $a'$ in the plane is a linear combination $(1-t)a + ta'$, for some real $t$, $0 \le t \le 1$. Summarizing the above algorithm, we have

LEMMA 3.3. *Given a half-length $\theta$-parallelogram $Q$ and the corresponding $\theta$-diameter, the next critical half-length parallelogram $Q^*$ and diameter can be computed in constant time. Furthermore, the set of half-length parallelograms for all intermediate angles can be computed as convex combinations of the corresponding endpoints of $Q$ and $Q^*$.*

Because each motion performed by the algorithm is locally counterclockwise we also have

(Interspersing Property). *As the angle $\theta$ rotates counterclockwise, the points of contact between the parallelogram move counterclockwise (or remain stationary) along the boundary of the $P$.*

As a consequence of the interspersing property we also have a bound on the number of critical events. Recall that by symmetry of half-length parallelograms, it suffices to consider rotating $\theta$ through $180^\circ$.

LEMMA 3.4. *Given a convex polygon $P$ with $n$ vertices, the number of critical angles is $3n$ over a rotation of* $180^\circ$.

PROOF. Let $\overrightarrow{a_1a_2}$ and $\overrightarrow{b_1b_2}$ denote the initial base chords. By Theorem 3.4, as $\theta$ rotates through $180^\circ$, the vertex $a_1$ will rotate monotonically counterclockwise to $b_2$, and $b_2$ will rotate monotonically counterclockwise to $a_1$. Thus, between these two vertices, exactly $n$ critical angles will be generated, corresponding to the moments at which these vectors become incident with vertices of $P$. Likewise, the two vertices $a_2$ and $b_1$ will generate a total of $n$ critical events. Finally each of the endpoints of the diameter chord in the direction $\theta$, will rotate counterclockwise about the boundary of $P$ until they exchange places. Thus, the number of event points generated by the diameter chord will also be $n$. This yields $3n$ total event points. □

From this last result and Lemma 3.3 we have the following corollary.

COROLLARY. *Given an $n$-sided convex polygon $P$, in $O(n)$ time we can compute the ordered sequence of half-length parallelograms at each of the critical angles from* $0^\circ$ *to* $180^\circ$.

**3.3. Minimizing between two critical angles.** To complete the algorithm it suffices to show how to compute the minimum area half-length parallelogram between two consecutive critical angles in constant time. Let $\theta$ and $\theta'$ be two angles such that there is no critical angle between them, and let $Q = \langle a_1, a_2, b_1, b_2\rangle$ and $Q' = \langle a_1', a_2', b_1', b_2'\rangle$ be the vertices of the corresponding half-length parallelograms, respectively. Because there are no critical angles between $\theta$ and $\theta'$, corresponding pairs vertices ($a_1$ and $a_1'$, for example) lie on the same edges of $P$. By Lemma 3.3 it follows that all half-length parallelograms between these two can be interpolated by considering convex combinations of corresponding vertices of $Q$ and $Q'$. For $i = 1, 2$ and $0 \le t \le 1$, define $a_i(t) = (1-t)a_i + ta_i'$ and $b_i(t) = (1-t)b_i + tb_i'$. Let us assume that the vertices of $Q$ are enumerated so that the chords $\overrightarrow{a_1a_2}$ and $\overrightarrow{b_1b_2}$ are parallel and similarly directed. Taking the point $b_1$ as the origin of the parallelogram, the vectors defining this parallelogram are $b_2(t) - b_1(t)$ and $a_1(t) - b_1(t)$. Thus the area of the interpolated parallelogram is given by the absolute value of the determinant

$$\det(b_2(t) - b_1(t),\ a_1(t) - b_1(t)).$$

This determinant is a polynomial of degree two in the invariant $t$, and this polynomial can be computed in $O(1)$ time given the endpoints of the two parallelograms $Q$ and $Q'$. Therefore, by differentiating the polynomial symbolically, we can determine the minima in the interval $[0, 1]$ in constant time.

The entire algorithm for determining the densest double-lattice packing of a convex polygon is given below. The correctness and $O(n)$ running time of this algorithm follow from the previous discussion.

ALGORITHM. (Finding the maximum density double-lattice packing):

(1) Compute an initial half-length parallelogram $Q$ for $\theta = 0$.
(2) While $\theta \leq 180^\circ$ do:
  (a) Advance to the next critical half-length parallelogram $Q'$ using Algorithm 3.2.
  (b) Determine the minimum area half-length parallelogram between $Q$ and $Q'$ by the method described above.
  (c) Let $Q := Q'$.
(3) Let $Q = \langle a_1, a_2, b_1, b_2 \rangle$ be the minimum area half-length parallelogram found in step (2b). Translate $Q$ so that $b_1$ coincides with the origin. Let $u = b_2 - b_1$ and let $v = a_1 - b_1$. Pack $P$ in the plane using the lattice generated by $2u$ and $2v$ and pack $-P$ using this same lattice.

## 4. Concluding remarks

We have shown that the problem of computing the densest double-lattice packing of a convex polygon is solvable in linear time. There are a number of interesting open problems suggested by this work. The first is to generalize the problem to the densest double-lattice packings of other types of shapes in the plane, for example, the connected union of two convex polygons, or star-shaped polygons. Unfortunately, the characterization of the densest packing in terms of extensive parallelograms relies heavily on convexity. A second problem is to consider other periodic structures analogous to the double-lattice for packings which allow rotation and which may be more economical for other types of convex polygons.

## REFERENCES

1. N. Adlai and A. DePano, *Polygon approximation with optimized polygonal enclosures: Applications and algorithms*, Ph.D. Thesis, Dept. of Computer Science, University of New Orleans, 1987.
2. I. Fáry, *Sur la densité des réseaux de domaines convexes*, Bull. Soc. Math. France **78** (1950), 152–161.
3. G. Fejes Tóth, *New results in the theory of packing and covering*, Convexity and Its Applications (P. M. Gruber and J. M. Wills, eds.), Birkhäuser Verlag, Basel, 1983, pp. 318–359.
4. L. Fejes Tóth, *One the densest packing of domain*, Proc. Kon. Ned. Akad. Wet. **51** (1948), 189–192.
5. M. R. Garey and D. S. Johnson, *Computers and intractibility: A guide to the theory of NP-completeness*, W. H. Freeman and Company, New York, 1979.
6. B. Grünbaum and G. C. Shephard, *Tilings and patterns*, W. H. Freeman and Company, New York, 1987.
7. H. Heesch, *Reguläres Parkettierungsproblem*, Westdeutscher Verlag, Cologne and Opladen, 1968.

8. V. Klee and M. L. Laskowski, *Finding the smallest triangles containing a given convex polygon*, J. Algorithms **6** (1985), 359–375.
9. G. Kuperberg and W. Kuperberg, *Double-lattice packings of convex bodies in the plane*, Discrete and Comput. Geom. **5** (1990), 389–397.
10. W. Kuperberg, *Covering the plane with congruent copies of a convex body*, Bull. London Math. Soc. **21** (1989), 82–86.
11. K. Mahler, *The theorem of Minkowski-Hlawka*, Duke Math. J. **13** (1946), 611–621.
12. D. M. Mount and R. Silverman, *Packing and covering the plane with translates of a convex polygon*, J. Algorithms **11** (1990), 564–580.
13. J. O'Rourke, A. Aggarwal, S. Maddila, and M. Baldwin, *An optimal algorithm for finding minimal enclosing triangles*, J. Algorithms **7** (1986), 258–269.
14. C. A. Rogers, *The closest packing of convex two-dimensional convex sets*, Acta Math. **86** (1951), 309–321.
15. C. A. Rogers, *Packing and covering*, Cambridge Univ. Press, London, 1964.
16. G. T. Toussaint, *Solving geometric problems with the 'rotating calipers,'* Proc. IEEE Melecon 83, 1983.

DEPARTMENT OF COMPUTER SCIENCE AND INSTITUTE FOR ADVANCED COMPUTER STUDIES, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742

# DIMACS

Series in Discrete Mathematics
and Theoretical Computer Science

Volume 6


# Discrete and Computational Geometry: Papers from the DIMACS Special Year


Jacob E. Goodman
Richard Pollack
William Steiger
Editors



NSF Science and Technology Center
in Discrete Mathematics and Theoretical Computer Science
A consortium of Rutgers University, Princeton University,
AT&T Bell Labs, Bellcore

**American Mathematical Society**
**Association for Computing Machinery**

This DIMACS volume on the Special Year in Discrete and Computational Geometry contains proceedings of workshops held at DIMACS from September 1989 through August 1990.

1991 *Mathematics Subject Classification*. Primary 03, 05, 12, 13, 14, 15, 32, 51, 52, 57, 68.

---

**Library of Congress Cataloging-in-Publication Data**

Discrete and computational geometry: papers from the DIMACS special year/Jacob E. Goodman, Richard Pollack, William Steiger, editors.
p. cm.—(DIMACS series in discrete mathematics and theoretical computer science; ISSN 1052-1798; v. 6)
Includes bibliographical references.
AMS: ISBN 0-8218-6595-1 (acid-free paper)
ACM: ISBN 0-89791-464-3
1. Geometry–Data processing–Congresses. I. Goodman, Jacob E. II. Pollack, Richard D. III. Steiger, William L., 1939- . IV. DIMACS (Group). V. American Mathematical Society. VI. Association for Computing Machinery. VII. Series.
QA448.D38D57 1991 91-36806
516′.00285-dc20 CIP

---

To order through AMS contact the AMS Customer Services Department, P.O. Box 6248, Providence, Rhode Island 02940-6248 USA. For VISA or MASTERCARD orders call 1-800-321-4AMS. Order code DIMACS/6.

To order through ACM contact ACM Order Department, P.O. Box 64145, Baltimore, Maryland 21264. Phone 301-528-4261. Order number 222915.

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy an article for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication (including abstracts) is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Manager of Editorial Services, American Mathematical Society, P.O. Box 6248, Providence, Rhode Island 02940-6248.

The appearance of the code on the first page of an article in this book indicates the copyright owner's consent for copying beyond that permitted by Sections 107 or 108 of the U.S. Copyright Law, provided that the fee of $1.00 plus $.25 per page for each copy be paid directly to the Copyright Clearance Center, Inc., 27 Congress Street, Salem, Massachusetts 01970. This consent does not extend to other kinds of copying, such as copying for general distribution, for advertising or promotional purposes, for creating new collective works, or for resale.

Copyright ©1991 by the American Mathematical Society. All rights reserved.
The American Mathematical Society retains all rights except those granted to the United States Government.

Printed in the United States of America.
The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability. ♾
This publication was typeset using AMS-TEX, the American Mathematical Society's TEX macro system.

10 9 8 7 6 5 4 3 2 1 95 94 93 92 91