Introduction to Game Theory

5. Lookahead Pathology

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Motivation

- When discussing game-tree search in the previous session, I said:
  - Deeper lookahead (i.e., larger depth bound \(d\)) usually gives better decisions

- For many years, it was tacitly assumed that searching deeper would always give better decisions
  - For my Ph.D. work in 1979, I showed that’s not true
  - There are infinitely many game trees for which searching deeper gives worse decisions
P-Games

- A class of board-splitting games invented by Judea Pearl in 1980
- Playing board: chessboard of size $2^{\lceil h/2 \rceil} \times 2^{\lceil h/2 \rceil}$ instead of $8 \times 8$
  - (or equivalently, a string of $2^h$ squares)
- Initial state: randomly label each square as “win” or “loss”
  - I’ll use green for win, white for loss
- Agents move in alternation
  - 1st move: remove either the left half or right half of the board
  - 2nd move: remove either the top half or bottom half of the board
- Continue until just one square is left
  - “win” square => win for the last player
  - “loss” square => loss for the last player
- This gives us a game tree of height $h$
**Critical Nodes**

- Let $x$ be a node in a P-game
  - Suppose $x$’s **height** (number of moves from the end of the game) is $h$
- In order to talk about whether a deeper search at $x$ gives a better or worse decision, $x$ must be a node where the decision makes a difference
  - $x$’s children shouldn’t have the same minimax value
- $x$ is **critical** if
  - it has a “loss” child $y$, i.e., $u^*(y) = -1$
  - and a “win” child $z$, i.e., $u^*(z) = 1$
- Let $D(d,h) = P(\text{choose the “win” child | minimax search to depth } d \text{ from a critical node } x \text{ of height } h)$
- Then $D(d,h) = P[\text{MINIMAX}(y,d-1) < \text{MINIMAX}(z,d-1)]$
  - $+ 0.5 \ P[\text{MINIMAX}(y,d-1) = \text{MINIMAX}(z,d-1)]$
  - where $y$ and $z$ are $x$’s loss child and win child
Probability of a Win Node

- Let $w = (3 - \sqrt{5})/2 \approx 0.382$
  - i.e., $w = 2 - \varphi = 1 - 1/\varphi$, where $\varphi$ is the golden ratio
- Suppose we assign a “win” or “loss” label to each square at random, with probability $p$ that a square is labeled “win”
- Let $x$ be a node of height $h$, and $y$ and $z$ be its children
  - If $p > w$, then as we increase $h$, $P[y \text{ and } z \text{ are both wins for the last player}] \to 1$
  - If $p < w$, then as we increase $h$, $P[y \text{ and } z \text{ are both losses for the last player}] \to 1$
  - If $p = w$, then for all $h$, $P[u^*(y) \neq u^*(z)] = p(1-p)$
- So from now on, let $p = w$
  - This assures a reasonably good chance that a node at height $h$ is critical
Evaluation Function

- Let $e(x) = \text{(number of “win” squares) / (total number of squares)}$
  - The higher $e(x)$ is, the more likely that $x$ is a win for the last player
  - The lower $e(x)$ is, the more likely that $x$ is a win for the other player
- Now that we have $e$, it’s possible to derive a formula for $D(d,h)$
  - The derivation is complicated and I’ll skip it
- But I’ll show you the results
P-Games are Pathological

- If $d = h$, then $D(d,h) = 1$
  - i.e., searching to the game’s end produces perfect play
- Likewise when $d = h-1$
  (searching to just before the end)
- For node height $h \leq 7$, no pathology
  - $D(d,h)$ generally increases as we increase $d$
- For node height $h > 9$, there’s lots of pathology
  - $D(d,h)$ generally decreases as we increase $d$
Why are the games pathological?

- **Hypothesis 1**: maybe it’s due to the evaluation function
  - Let the **height** of a node be its distance from the end of the game
  - At a node of height $h$, a depth-$d$ minimax search will apply the evaluation function $e$ to nodes of height $h-d$
    - Increase the search depth $d \Rightarrow$ decrease the node height $h-d$
    - If $e$ is less accurate at nodes whose height is low, this could make $D(d,h)$ decrease as we increase $d$
  - To find out, let’s measure $e$’s accuracy as a function of node height
    - $e$’s accuracy at a critical node $x$ of height $h$
      $= P[\text{correct decision if we apply } e \text{ directly to } x\text{'s children}]$
      $= D(1,h)$
  - So let’s look at $D(1,h)$ as $h \rightarrow 0$
Why are the games pathological?

- The graph shows $D(1,h)$ as a function of $h$
- Notice that as $h \to 0$, $D(1,h) \to 1$
  - I.e., as $x$’s height decreases, $e(x)$ gets more accurate
- Thus the hypothesis is wrong
  - The pathology isn’t due to the evaluation function
  - It must be due to the game itself
Why are the games pathological?

- Hypothesis 2:
  - In most board games,
    - Some positions are “strong” (you’re likely to win)
    - Others are “weak” (you’re likely to lose)
    - Strong nodes are likely to have lots of strong children
      - So if a node is strong, that means its sibling nodes are probably strong too
    - Likewise for weak positions
  - But in P-games, the values of sibling nodes are completely independent of each other
    - Could the pathology be due to that?
  - Let’s modify P-games to make sibling nodes have similar values
**N-Games**

- Everything is the same as in a P-game, except for how the board is initialized:
  - First assign 1 or –1 at random to each edge of the game tree
  - A node $x$’s “strength” = sum of the edges on the path from the root to $x$
  - If $x$ is a terminal node,
    - Label $x$ “win” if $\text{strength}(x) > 0$
    - Otherwise label $x$ “loss”
- Use the same evaluation function as before
N-Games

- I don’t know of a formula for computing $D(d,h)$ in N-games
  - So, Monte Carlo simulation instead
- For every combination of node height $h$ and search depth $d$, I averaged $D(d,h)$ over 3200 randomly generated N-games
  - Result: at every node height $h$, searching deeper always helps
- So this suggests pathology is unlikely when there’s a strong local similarity (correlation among sibling nodes)
Generalize to Other Games

- Suppose we do a minimax search to depth 2 at node $a$
  - $e$ and $h$ look equally good, and both look better than $b$
  - So we choose one of $e$ and $h$ at random, and move to it

- What’s the probability that we made a best move?
Probability of Optimal Decision

- For every node $x$, let $s(x) = \{x’s \text{ children}\}$
- Let $\text{opt}(x,d) = \{\text{the children of } x \text{ that look best to a depth-}d \text{ minimax search}\}$
  - $= \{y \in s(x) | \text{minimax}(x,d) = \text{minimax}(y,d-1)\}$
  - In the example, $\text{opt}(a,2) = \{e,h\}$
- The children of $x$ that really are the best are the ones in $\text{opt}(x,\infty)$
  - I.e., search to the end of the game
  - In the example, $\text{opt}(a,\infty) = \{e\}$
- If we choose from $\text{opt}(x,d)$ at random, then the probability of choosing an optimal move is
  - $P_{\text{opt}}(x,d) = |\text{opt}(x,d) \cap \text{opt}(x,\infty)| / |\text{opt}(x,d)|$
- In the example, $P_{\text{opt}}(a,2) = |\{e\}| / |\{e,h\}| = \frac{1}{2}$
Degree of Pathology

• The **decision error** at $x$ is the probability that we didn’t make the best choice:
  $$P_{err}(x,d) = 1 - P_{opt}(x,d)$$

• The **degree of pathology** at $x$ is the probability that searching deeper increases the decision error:
  $$p(x,i,j) = P_{err}(x,i) / P_{err}(x,j)$$
  where $i$ and $j$ are search depths, and $i > j$

• If $p(x,i,j) > 1$ then we have lookahead pathology at $x$

• A game $G$ is considered **pathological** if $p(x,i,j)$, averaged over many $x$, is $> 1$
  - When $G$ is pathological for some values of $i$ and $j$, it usually is pathological for others
Influences on the Degree of Pathology

- Several factors affect the degree of pathology
- The most important ones:
  - **Granularity**
    - Number of possible utility values
  - **Branching factor**
    - Number of children of each node
  - **Local similarity**
    - Similarity among nodes that are close together in the tree
- There are several others
  - But most of them reduce to special cases of the ones above
How to Vary the Branching Factor

- Easy to get P-games and N-games of branching factor $b$
  - The board has size $b^{\lceil h/2 \rceil} \times b^{\lfloor h/2 \rfloor}$
    - (or equivalently, a string of $b^h$ squares)
  - Each move: divide the board into $b$ pieces instead of 2 pieces, and discard all but one of them
- Result: a $b$-ary tree of height $h$
How to Vary the Granularity

- **P-game with infinite granularity:**
  - each square isn’t “win” or “loss”
  - instead, its payoff is uniformly distributed over $[0,1]$

- **N-game with infinite granularity:**
  - Instead of assigning 1 or –1 to each edge, assign a random value from a normal (i.e., Gaussian) distribution

- **P-game or N-game with granularity $g$:**
  - Partition the interval $[0,1]$ into $g$ intervals of equal size
How to Vary the Local Similarity

- Use a parameter $0 \leq s \leq 1$ that determines the amount of local similarity:
  - $s = 0 \Rightarrow$ P-game of granularity $g$
  - $s = 1 \Rightarrow$ N-game of granularity $g$
  - $0 < s < 1 \Rightarrow$
    - Generate both P-game and N-game values for the nodes
    - For each terminal node, assign a payoff by making a random choice:
      - The node’s P-game value with probability $s$, or its N-game value with probability $1-s$
Evaluation Function and Experiments

- So now we can vary $b$, $g$, and $s$ independently
  - Experiments to measure how they influence the degree of pathology

- We can’t use the previous evaluation function
  - It only works when $g = 2$

- Instead, use the following:
  - $e(x) = x$’s actual minimax value, corrupted by Gaussian noise with standard deviation $\sigma = 0.1$
  - For this evaluation function, accuracy is independent of node height

Granularity and Pathology

- Amount of granularity needed to avoid lookahead pathology
  - The space above the surface is pathological
  - The space below the surface is nonpathological
Branching Factor and Pathology

- The degree of pathology as a function of branching factor, granularity, and local similarity
  - Color of each point = value of $p(5,1)$
- Below the black lines: pathological
- Above the black lines: nonpathological.
Does the Model Have Predictive Value?

- Does the model predict the trends in real games?
  - Yes!

- Let’s look at
  - chess
  - kalah
Chess endgames

- Degree of pathology as a function of granularity in
  - KBBK chess endgames (average $b = 13.52$ and $cf = 0.58$)
  - KQKR chess endgames (average $b = 16.93$ and $cf = 0.37$)
Kalah

- An ancient African game
- Moves:
  - Pick up the seeds in a pit on your side of the board
  - Distribute them, one at a time, to a string of adjacent pits
- Objective: acquire more seeds than the opponent, by either
  - moving them to your “kalah”
  - capturing them from the opponent’s pits
Modified Kalah

- Kalah is normally played until no seeds are left on the board
  - For computability, we limited the game to 8 moves
- To ensure a uniform branching factor
  - We allowed players to “move” from an empty pit
  - Such a move has no effect on the board
- We got different branching factors by varying the number of pits
- In Kalah, a player can move again if the last seed they placed lands in their kalah
  - We eliminated that rule, to get strict alternation of moves
Modified Kalah

- Degree of pathology in modified kalah as a function of granularity for several different branching factors
Modified Kalah

- The degree of pathology in modified kalah at several different branching factors, as a function of clustering factor (cf)

\[ \text{standard deviation of the sibling nodes’ utilities} \div \text{standard deviation of the utilities throughout the game tree} \]

- Higher cf means less local similarity

- Curves are smoothed for clarity
Summary

- In most game trees
  - Increasing the search depth usually improves the decision-making
- In pathological game trees
  - Increasing the search depth usually degrades the decision-making
- Pathology is more likely when
  - The branching factor is high
  - The number of possible payoffs is small
  - Local similarity is low
- Even in ordinary non-pathological game trees, *local* pathologies can occur
  - Work in progress: some of my students are developing algorithms to detect and overcome local pathologies