1 Introduction

How does one prove that all DFA’s recognize CFL’s. Here are two ways. Let $M$ be a DFA.

1. Since all DFA’s are PDA’s, $M$ is a PDA. For all PDA’s $M$ there exists CFL $G$ such that $L(M) = L(G)$. The drawback of this proof is that it requires PDA-to-CFG theorem.

2. For all DFA’s $M$ there exists a regular expression $\alpha$ such that $L(M) = L(\alpha)$. By induction on the formation of a regular expression one can easily show that, for all regular expressions $\alpha$, $L(\alpha)$ is a CFL. The drawback of this proof is that it requires the DFA-to-Reg Expression theorem.

Can one transform a DFA into a CFG directly? Yes. We give the proof in the next section. It resembles the proof that if $M$ is DFA then there is a regular expression for $\alpha$ such that $L(M) = L(\alpha)$. (The $R(i, j, k)$ construction.)

2 The Direct Proof

**Theorem** If $M$ is a DFA then there exists CFG $G$ such that $L(M) = L(G)$.

**Proof:** Let $M = (Q, \Sigma, \delta, s, F)$. We can assume $Q = \{1, \ldots, n\}$ and that $s = 1$. We want to form CFG

$G = (N, \Sigma, P, S)$ such that $L(G) = L(M)$.

- $N = \{[i, j, k] : (i, j, k) \in Q \times Q \times \{0, \ldots, n\}\} \cup \{S\} \cup \{[k, k-k-1, :] : k \in \{2, \ldots, n\}\}$.

  Our intention is that $[i, j, k] \Rightarrow \{w \in \Sigma^* : \bar{\delta}(i, w) = j \text{ and the path only uses states } \leq k\}$.

- $S$ is the start state.

- We define $P$ the set of productions.
  - For every $f \in F$ we have the production $S \rightarrow (1, f, n)$. 

- **Generating base productions.** For every $i, j \in Q$,
  
  $[i, j, 0] \rightarrow \{ \sigma \in \Sigma : \delta(i, \sigma) = j \}$
  
  $[i, i, 0] \rightarrow e$

- **Generating the rest of the productions.** For every $i, j \in Q$, and every $1 \leq k \leq n$:

  $[i, j, k] \rightarrow [i, j, k-1]$

  $[i, j, k] \rightarrow [i, k, k-1][k, k, k-1, *][k, j, k-1]$

  $[k, k, k-1, *] \rightarrow [k, k, k-1][k, k, k-1, *]$

  $[k, k, k-1, *] \rightarrow e$

The reader can show by induction on $k$ that, for all $k, i, j,

$[i, j, k] \Rightarrow \{ w \in \Sigma^* : \delta(i, w) = j \text{ and the path only uses states } \leq k \}$. 

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