Quantum algorithms (CO 781/CS 867/QIC 823, Winter 2011)
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LECTURE 2: The abelian QFT and phase estimation

Quantum Fourier transform

Perhaps the most important unitary transformation in quantum computing is the *quantum Fourier transform* (QFT). Later, we will discuss the QFT over arbitrary finite groups, but for now we will focus on the case of an abelian group G. Here the transformation is

$$F_G := \frac{1}{\sqrt{|G|}} \sum_{x \in G} \sum_{y \in \hat{G}} \chi_y(x) |y\rangle\langle x| \tag{1}$$

where $\hat{G}$ is a complete set of characters of G, and $\chi_y(x)$ denotes the yth character of G evaluated at x. (You can verify that this is a unitary operator using the orthogonality of characters.) Since G and $\hat{G}$ are isomorphic, we can label the elements of $\hat{G}$ using elements of G, and it is often useful to do so.

The simplest QFT over a family of groups is the QFT over $G = \mathbb{Z}_2^n$. The characters of this group are $\chi_y(x) = (-1)^{x \cdot y}$, so the QFT is simply

$$F_{\mathbb{Z}_2^n} = \frac{1}{\sqrt{2^n}} \sum_{x,y \in \mathbb{Z}_2^n} (-1)^{x \cdot y} |y\rangle\langle x| = H^{\otimes n}. \tag{2}$$

You have presumably seen how this transformation is used in the solution of Simon's problem.

QFT over $\mathbb{Z}_{2^n}$

A more complex quantum Fourier transform is the QFT over $G = \mathbb{Z}_{2^n}$:

$$F_{\mathbb{Z}_{2^n}} = \frac{1}{\sqrt{2^n}} \sum_{x,y \in \mathbb{Z}_{2^n}} \omega_{2^n}^{xy} |y\rangle\langle x| \tag{3}$$

where $\omega_m := \exp(2\pi \mathrm{i}/m)$ is a primitive mth root of unity. To see how to realize this transformation by a quantum circuit, it is helpful to represent the input x as a string of bits, $x = x_{n-1} \dots x_1 x_0$, and to consider how an input basis vector is transformed:

$$\begin{aligned}
|x\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{y \in \mathbb{Z}_{2^n}} \omega_{2^n}^{xy} |y\rangle & \qquad (4)\\
= \frac{1}{\sqrt{2^n}} \sum_{y \in \mathbb{Z}_{2^n}} \omega_{2^n}^{x(\sum_{k=0}^{n-1} y_k 2^k)} |y_{n-1} \dots y_1 y_0\rangle & \qquad (5)\\
= \frac{1}{\sqrt{2^n}} \sum_{y \in \mathbb{Z}_{2^n}} \prod_{k=0}^{n-1} \omega_{2^n}^{x y_k 2^k} |y_{n-1} \dots y_1 y_0\rangle & \qquad (6)\\
= \frac{1}{\sqrt{2^n}} \bigotimes_{k=0}^{n-1} \sum_{y_k \in \mathbb{Z}_2} \omega_{2^n}^{x y_k 2^k} |y_k\rangle & \qquad (7)\\
= \bigotimes_{k=0}^{n-1} |z_k\rangle & \qquad (8)
\end{aligned}$$

where

$$|z_k\rangle := \frac{1}{\sqrt{2}} \sum_{y_k \in \mathbb{Z}_2} \omega_{2^n}^{x y_k 2^k} |y_k\rangle \tag{9}$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + \omega_{2^n}^{x 2^k} |1\rangle) \tag{10}$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + \omega_{2^n}^{\sum_{j=0}^{n-1} x_j 2^{j+k}} |1\rangle) \tag{11}$$

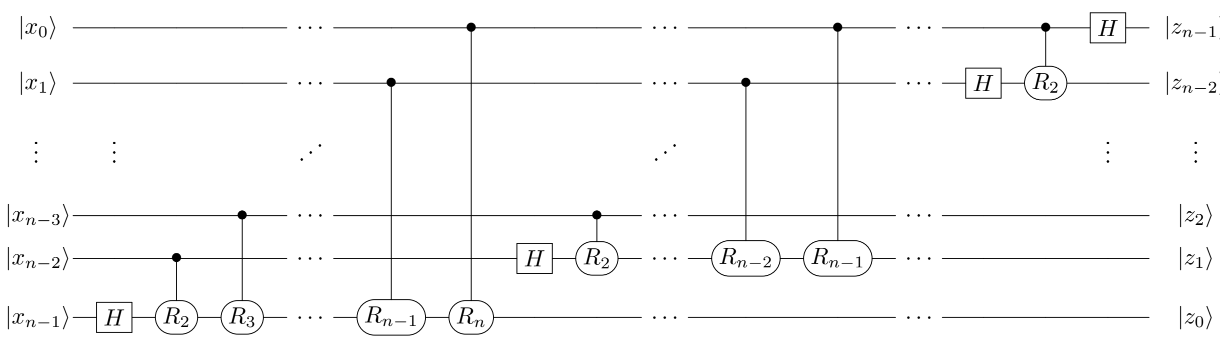
$$= \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi \mathrm{i} (x_0 2^{k-n} + x_1 2^{k+1-n} + \cdots + x_{n-1-k} 2^{-1})} |1\rangle). \tag{12}$$

(A more succinct way to write this is $|z_k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \omega_{2^{n-k}}^x |1\rangle)$, but the above expression is more helpful for understanding the circuit.) In other words, $F|x\rangle$ is a tensor product of single-qubit states, where the kth qubit only depends on the k least significant bits of x.

This decomposition immediately gives a circuit for the QFT over $\mathbb{Z}_{2^n}$. Let R_k denote the single-qubit unitary operator

$$R_k := \begin{pmatrix} 1 & 0 \\ 0 & \omega_{2^k} \end{pmatrix}. \tag{13}$$

Then the circuit can be written as follows:



This circuit uses $O(n^2)$ gates. However, there are many rotations by small angles that do not affect the final result very much. If we simply omit the gates R_k with $k = \Omega(\log n)$, then we obtain a circuit with $O(n \log n)$ gates that implements the QFT with precision $1/\operatorname{poly}(n)$.

Phase estimation

Aside from being directly useful in quantum algorithms, such as Shor's algorithm, The QFT over $\mathbb{Z}_{2^n}$ provides a useful quantum computing primitive called *phase estimation.* In the phase estimation problem, we are given a unitary operator U (either as an explicit circuit, or as a black box that lets us apply a controlled-U^j operation for integer values of j). We are also given a state $|\phi\rangle$ that is promised to be an eigenvector of U, namely $U|\phi\rangle = e^{\mathrm{i}\phi}|\phi\rangle$ for some $\phi \in \mathbb{R}$. The goal is to output an estimate of ϕ to some desired precision.

The procedure for phase estimation is straightforward. To get an n-bit estimate of ϕ, prepare the quantum computer in the state

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{Z}_{2^n}} |x, \phi\rangle, \tag{14}$$

apply the operator

$$\sum_{x \in \mathbb{Z}_{2^n}} |x\rangle\langle x| \otimes U^x \tag{15}$$

to give the state

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{Z}_{2^n}} e^{\mathrm{i}\phi x} |x, \phi\rangle, \tag{16}$$

apply an inverse Fourier transform on the first register, and measure. If the binary expansion of $\phi/2\pi$ terminates after at most n bits (i.e., if $\phi = 2\pi y/2^n$ for some $y \in \mathbb{Z}_{2^n}$), then the state (16) is $F_{2^n}|y\rangle \otimes |\phi\rangle$, so the result is guaranteed to be the binary expansion of $\phi/2\pi$. In general, we obtain a good approximation with high probability. (You have probably seen this kind of calculation before, and we will see the details of a similar calculation when we discuss period finding.)

QFT over $\mathbb{Z}_N$ and over a general finite abelian group

One useful application of phase estimation is to implement the QFT over an arbitrary cyclic group $\mathbb{Z}_N$:

$$F_{\mathbb{Z}_N} = \frac{1}{\sqrt{N}} \sum_{x,y \in \mathbb{Z}_N} \omega_N^{xy} |y\rangle\langle x|. \tag{17}$$

The circuit we derived using the binary representation of the input and output only works when N is a power of two (or, with a slight generalization, some other small integer). But there is a simple way to realize $F_{\mathbb{Z}_N}$ (approximately) using phase estimation.

We would like to perform the transformation that maps $|x\rangle \mapsto |\tilde{x}\rangle$, where $|\tilde{x}\rangle := F_{\mathbb{Z}_N}|x\rangle$ denotes a Fourier basis state. (By linearity, if the transformation acts correctly on a basis, it acts correctly on all states.) It is straightforward to perform the transformation $|x, 0\rangle \mapsto |x, \tilde{x}\rangle$; then it remains to erase the register $|x\rangle$ from such a state.

Consider the unitary operator that adds 1 modulo N:

$$U := \sum_{x \in \mathbb{Z}_N} |x+1\rangle\langle x|. \tag{18}$$

The eigenstates of this operator are precisely the Fourier basis states $|\tilde{x}\rangle := F_{\mathbb{Z}_N}|x\rangle$, since (as a simple calculation shows)

$$F_{\mathbb{Z}_N}^\dagger U F_{\mathbb{Z}_N} = \sum_{x \in \mathbb{Z}_N} \omega_N^x |x\rangle\langle x|. \tag{19}$$

Thus, using phase estimation on U (with n bits of precision where $n = O(\log N)$), we can perform the transformation

$$|\tilde{x}, 0\rangle \mapsto |\tilde{x}, x\rangle \tag{20}$$

(actually, phase estimation only gives an approximation of x, so we implement this transformation only approximately). By running this operation in reverse, we can erase $|x\rangle$, and thereby produce the desired QFT.

Given the Fourier transform over $\mathbb{Z}_N$, it is straightforward to implement the QFT over an arbitrary finite abelian group: any finite abelian group can be written as a direct product of cyclic factors, and the QFT over a direct product of groups is simply the tensor product of QFTs over the individual groups.