Examples of Three Person Cake Cutting With Uniform Valuations

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Credit Where Credit is Due

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The paper

How to Cut a Cake Before the Party Ends

by

David Kurokawa, John K. Lai, Ariel Procaccia

has a protocol for envy-free cake cutting with piecewise linear valuations. Their paper inspired these slides.

We refer to their paper as ENDS.

Alice, Bob, Carol

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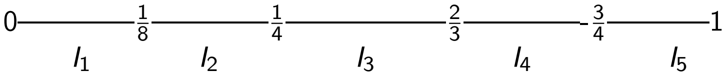
Alice's tastes are uniform on $[\frac{1}{8}, 1]$. Multiplier: $\frac{8}{7}$.

Bob's tastes are uniform on $[0, \frac{2}{3}]$. Multiplier: $\frac{3}{2}$.

Carol's tastes are uniform on $[\frac{1}{4}, \frac{3}{4}]$. Multiplier: 2.

Intervals

Intervals



- How much of I_1 should Alice get?
- How much of I_1 should Bob get?
- How much of I_1 should Carol get?
- How much of I_2 should Alice get?
- How much of I_2 should Bob get?
- How much of I_2 should Carol get?
- Etc.

Variables

Variables

x_{1A} is how much Alice gets of I_1.
x_{1B} is how much Bob gets of I_1.
x_{1C} is how much Carol gets of I_1.
x_{2A} is how much Alice gets of I_2.
x_{2B} is how much Bob gets of I_2.
x_{2C} is how much Carol gets of I_2.
$\vdots$
x_{iP} is how much Person P gets of I_i.

NOTE: $x_{1A} = x_{4B} = x_{5B} = x_{1C} = x_{2C} = x_{5C} = 0$.

Example: $x_{2A} = \frac{1}{10} \rightarrow$ Alice gets subinterval of I_2 of length $\frac{1}{10}$.

Equations: The x_{iP} Make Sense

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I_1 of length $\frac{1}{8}$: $0 \le x_{1B} \le \frac{1}{8} - 0 = \frac{1}{8}$

I_2 of length $\frac{1}{8}$: $0 \le x_{2A}, x_{2B} \le \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$.

I_3 of length $\frac{5}{12}$: $0 \le x_{3A}, x_{3B}, x_{3C} \le \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$

I_4 of length $\frac{1}{12}$: $0 \le x_{4A}, x_{4C} \le \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$

I_5 of length $\frac{1}{4}$: $0 \le x_{5A} \le 1 - \frac{3}{4} = \frac{1}{4}$

We will not mention these again for a while.

Equations: The x_{iP} Make Sense

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I_1 of length $\frac{1}{8}$: $\quad x_{1B} = \frac{1}{8}$

I_2 of length $\frac{1}{8}$: $\quad x_{2A} + x_{2B} = \frac{1}{8}$

I_3 of length $\frac{5}{12}$: $\quad x_{3A} + x_{3B} + x_{3C} = \frac{5}{12}$

I_4 of length $\frac{1}{12}$: $\quad x_{4A} + x_{4C} = \frac{1}{12}$

I_5 of length $\frac{1}{4}$: $\quad x_{5A} = \frac{1}{4}$

We set

$$x_{1B} = \frac{1}{8} \qquad x_{5A} = \frac{1}{4}.$$

The first and fifth equation are now satisfied.

Equations: Getting Everyone $\geq \frac{1}{3}$

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Alice gets $\geq \frac{1}{3}$: $\frac{8}{7}(x_{2A} + x_{3A} + x_{4A} + \frac{1}{4}) \geq \frac{1}{3}$

$$\frac{8}{7}(x_{2A} + x_{3A} + x_{4A}) \geq \frac{1}{21}$$

Bob gets $\geq \frac{1}{3}$: $\frac{3}{2}(\frac{1}{8} + x_{2B} + x_{3B}) \geq \frac{1}{3}$

$$\frac{3}{2}(x_{2B} + x_{3B}) \geq \frac{7}{48}$$

Carol gets $\geq \frac{1}{3}$:

$$2(x_{3C} + x_{4C}) \geq \frac{1}{3}$$

ALL the Equations

ALL the Equations

All vars ≥ 0.

$$\begin{aligned} x_{2A} + x_{2B} &= \frac{1}{8} \\ x_{3A} + x_{3B} + x_{3C} &= \frac{5}{12} \\ x_{4A} + x_{4C} &= \frac{1}{12} \\ \frac{8}{7}(x_{2A} + x_{3A} + x_{4A}) &\geq \frac{1}{21} \\ \frac{3}{2}(x_{2B} + x_{3B}) &\geq \frac{7}{48} \\ 2(x_{3C} + x_{4C}) &\geq \frac{1}{3} \end{aligned}$$

Can solve by REASONING or by an LP package.

Reasoning- Carol First

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Reasoning:

- Give Carol first– she has largest multiplier.
- Give Carol from I_4, only Alice competes there.
- Giver her ALL of I_4 since still does not get Carol $\frac{1}{3}$.
- Recall:

$$\begin{aligned} x_{4A} + x_{4C} &= \frac{1}{12} \\ 2(x_{3C} + x_{4C}) &\geq \frac{1}{3} \end{aligned}$$

- Set $x_{4C} = \frac{1}{12}$. Forces $x_{4A} = 0$.
- $2(x_{3C} + \frac{1}{12}) \geq \frac{1}{3}$
- Set $x_{3C} = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$.
- Carol has $1/3$, Interval I_4 is allocated.

Making Bob Happy

Making Bob Happy

Plugging in $x_{4A} = 0$, $x_{4C} = \frac{1}{12}$, $x_{3C} = \frac{1}{12}$ yields:

$$\begin{aligned} x_{2A} + x_{2B} &= \frac{1}{8} \\ x_{3A} + x_{3B} &= \frac{1}{3} \\ \frac{8}{7}(x_{2A} + x_{3A}) &\geq \frac{1}{21} \\ \frac{3}{2}(x_{2B} + x_{3B}) &\geq \frac{7}{48} \end{aligned}$$

Satisfy Bob: Give Bob from smaller interval I_2 (makes math easier) give him ALL of it: $x_{2B} = \frac{1}{8}$. Forces $x_{2A} = 0$.

Making Bob Happy

Making Bob Happy

Plug in $x_{2B} = \frac{1}{8}$ and $x_{2A} = 0$.

$$\begin{aligned} x_{3A} + x_{3B} &= \tfrac{1}{3} \\ \tfrac{8}{7}(x_{3A}) &\geq \tfrac{1}{21} \\ \tfrac{3}{2}\left(\tfrac{1}{8} + x_{3B}\right) &\geq \tfrac{7}{48} \end{aligned}$$

Give Bob enough of l_2 so that he is happy:

$$\frac{1}{8} + x_{3B} \geq \frac{7}{72}$$

$$x_{3B} \geq \frac{55}{576}$$

Set $x_{3B} = \frac{55}{576}$. Forces $x_{3A} = \frac{1}{3} - \frac{55}{576} = \frac{137}{576}$. Does this work?

Final Reckoning

Final Reckoning

Alice: $x_{1A} = 0$, $x_{2A} = 0$, $x_{3A} = \frac{137}{576}$, $x_{4A} = 0$, $x_{5A} = \frac{1}{4}$.

$$\frac{8}{7}(0 + 0 + \frac{137}{576} + 0 + \frac{1}{4}) \sim 0.5575$$

Bob: $x_{1B} = \frac{1}{8}$, $x_{2B} = \frac{1}{8}$, $x_{3B} = \frac{55}{576}$, $x_{4B} = 0$, $x_{5B} = 0$.

$$\frac{3}{2}(\frac{1}{8} + 0 + \frac{1}{8} + \frac{55}{576} + 0 + 0) \sim 0.5182$$

Carol: $x_{1C} = 0$, $x_{2C} = 0$, $x_{3C} = \frac{1}{12}$, $x_{4C} = \frac{1}{12}$, $x_{5C} = 0$.

$$2(0 + 0 + \frac{1}{12} + \frac{1}{12} + 0) = \frac{1}{3} \sim 0.3333$$

TOTAL:

$$0.5575 + 0.5182 + 0.3333 = 1.409$$

MOST UNHAPPY: Carol with 0.33333.

Linear Programming

Linear Programming

The Linear Programming Problem Maximize (or Minimize) a LINEAR function relative to LINEAR constraints.

Example

Maximize

$$4x + 8y - 7z$$

Relative to

$$\begin{aligned} -3x + 5y - 8z &\leq 20 \\ x + y + z &\leq 5 \\ 2x + y + 18z &\leq 100 \\ 7x + 29y + 178z &\leq 193 \end{aligned}$$

- VERY practical problem. Many REAL applications.
- There are MANY PACKAGE for it that are easy to use: `http://www3.nd.edu/~jeff/mathprog/mathprog.html`

Linear Programming

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We want $x_{2A}, x_{2B}, x_{3A}, x_{3B}, x_{3C}, x_{4A}, x_{4C}$ that satisfies:

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$0 \leq x_{2A}, x_{2B} \leq \frac{1}{8}$

Linear Programming

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$0 \leq x_{2A}, x_{2B} \leq \frac{1}{8}$
$0 \leq x_{3A}, x_{3B}, x_{3C} \leq \frac{5}{12}$
$0 \leq x_{4A}, x_{4C} \leq \frac{1}{12}$

$x_{2A} + x_{2B} = \frac{1}{8}$

Linear Programming

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$x_{2A} + x_{2B} = \frac{1}{8}$

$x_{3A} + x_{3B} + x_{3C} = \frac{5}{12}$

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$x_{4A} + x_{4C} = \frac{1}{12}$

$\frac{8}{7}(x_{2A} + x_{3A} + x_{4A} + \frac{1}{4}) \geq \frac{1}{3}$

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Linear Programming

We want $x_{2A}, x_{2B}, x_{3A}, x_{3B}, x_{3C}, x_{4A}, x_{4C}$ that satisfies:

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$x_{2A} + x_{2B} = \frac{1}{8}$
$x_{3A} + x_{3B} + x_{3C} = \frac{5}{12}$
$x_{4A} + x_{4C} = \frac{1}{12}$

$\frac{8}{7}(x_{2A} + x_{3A} + x_{4A} + \frac{1}{4}) \geq \frac{1}{3}$
$\frac{3}{2}(\frac{1}{8} + x_{2B} + x_{3B}) \geq \frac{1}{3}$
$2(x_{3C} + x_{4C}) \geq \frac{1}{3}$

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Our Goal is WEAKER than Linear Programming- all we want to do is find SOME point.

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$$\frac{8}{7}(x_{2A} + x_{3A} + x_{4A} + \frac{1}{4}) + \frac{3}{2}(\frac{1}{8} + x_{2B} + x_{3B}) + 2(x_{3C} + x_{4C})$$

Maximizing Total Happiness

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Plugged into an LP package:

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Plugged into an LP package:

A: $x_{1A} = 0$, $x_{2A} = 0.0277$, $x_{3A} = 0.0138$, $x_{4A} = 0$. $x_{5A} = 0.25$

$$\frac{8}{7}(0 + 0.0277 + 0.0138 + 0 + 0.25) = 0.333$$

Maximizing Total Happiness

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B: $x_{1B} = 0.125$, $x_{2B} = 0.0972$, $x_{3B} = 0$, $x_{4B} = 0$, $x_{5B} = 0$.

$$\frac{3}{2}(0.125 + 0.0972 + 0 + 0 = 0) = 0.333$$

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$$\frac{3}{2}(0.125 + 0.0972 + 0 + 0 = 0) = 0.333$$

C: $x_{1C} = 0$, $x_{2C} = 0$, $x_{3C} = 0.403$, $x_{4C} = 0.083$, $x_{5C} = 0$.

$$2(0 + 0 + 0.403 + 0.083 + 0) = 0.972$$

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TOTAL:

$$0.3333 + 0.3333 + 0.97222 = 1.638$$

Maximizing Total Happiness

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$$2(0 + 0 + 0.403 + 0.083 + 0) = 0.972$$

TOTAL:

$$0.3333 + 0.3333 + 0.97222 = 1.638$$

MOST UNHAPPY: Alice and Bob 0.3333.

Minimize Unhappiness

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Add a variable t.

$\frac{8}{7}(x_{2A} + x_{3A} + x_{4A} + \frac{1}{4}) \geq t$

$\frac{3}{2}(\frac{1}{8} + x_{2B} + x_{3B}) \geq t$

$2(x_{3C} + x_{4C}) \geq t$

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$2(x_{3C} + x_{4C}) \geq t$

Maximize t.

Minimizing Ind. Unhappiness

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Plugged into an LP package:

Minimizing Ind. Unhappiness

Plugged into an LP package:

A: $x_{1A} = 0$, $x_{2A} = 0$, $x_{3A} = 0.17857$, $x_{4A} = 0$. $x_{5A} = 0.25$

$$\frac{8}{7}(0 + 0 + .178587 + 0.25) = 0.4898$$

Minimizing Ind. Unhappiness

Plugged into an LP package:

A: $x_{1A} = 0$, $x_{2A} = 0$, $x_{3A} = 0.17857$, $x_{4A} = 0$. $x_{5A} = 0.25$

$$\frac{8}{7}(0 + 0 + .178587 + 0.25) = 0.4898$$

B: $x_{1B} = 0.125$, $x_{2B} = 0.125$, $x_{3B} = 0.076531$, $x_{4B} = 0$, $x_{5B} = 0$.

$$\frac{3}{2}(0.125 + 0.125 + 0.076531 + 0 + 0) = 0.4898$$

Minimizing Ind. Unhappiness

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$$\frac{8}{7}(0 + 0 + .178587 + 0.25) = 0.4898$$

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$$\frac{3}{2}(0.125 + 0.125 + 0.076531 + 0 + 0) = 0.4898$$

C: $x_{1C} = 0$, $x_{2C} = 0$, $x_{3C} = 0.16156$, $x_{4C} = 0.083$, $x_{5C} = 0$.

$$2(0 + 0 + 0.16156 + 0.083 + 0) = 0.4898.$$

Minimizing Ind. Unhappiness

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TOTAL:

Minimizing Ind. Unhappiness

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$$\frac{3}{2}(0.125 + 0.125 + 0.076531 + 0 + 0) = 0.4898$$

C: $x_{1C} = 0$, $x_{2C} = 0$, $x_{3C} = 0.16156$, $x_{4C} = 0.083$, $x_{5C} = 0$.

$$2(0 + 0 + 0.16156 + 0.083 + 0) = 0.4898.$$

TOTAL:

$$0.4898 + 0.4898 + 0.4898 = 1.4694$$

Minimizing Ind. Unhappiness

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A: $x_{1A} = 0$, $x_{2A} = 0$, $x_{3A} = 0.17857$, $x_{4A} = 0$. $x_{5A} = 0.25$

$$\frac{8}{7}(0 + 0 + .178587 + 0.25) = 0.4898$$

B: $x_{1B} = 0.125$, $x_{2B} = 0.125$, $x_{3B} = 0.076531$, $x_{4B} = 0$, $x_{5B} = 0$.

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C: $x_{1C} = 0$, $x_{2C} = 0$, $x_{3C} = 0.16156$, $x_{4C} = 0.083$, $x_{5C} = 0$.

$$2(0 + 0 + 0.16156 + 0.083 + 0) = 0.4898.$$

TOTAL:

$$0.4898 + 0.4898 + 0.4898 = 1.4694$$

MOST UNHAPPY: ALL have 0.4898.

Protocol

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Protocol for n players, all have uniform valuations.

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Protocol

Protocol for n players, all have uniform valuations.

1) Every player simul reveals their valuation. (honestly)

2) Players form LP program to satisfy that all have $\ge 1/n$, vars make sense, and total is maximized (OR to minimize Unhappiness). They solve the LP.

3) Player make the cuts as the LP solution dictates.

- How many cuts? $\le 2n-1$ intervals, $\le n-1$ cuts. PLUS the cuts at each interval, $\le 2n-2$ cuts. TOTAL NUMBER OF CUTS: $\le (2n-1)(n-1)+2n-2 = 2n^2-n-2$.
- Does this LP always have a solution? Yes.
- The paper ENDS has an $O(n^2)$ protocol for envy-free (hence prop) but does not maximize total. Extends to piece-wise valuations but with diff bound depending on number-of-pieces.

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Alice not envious of Bob: $x_{2A} + x_{3A} + x_{4A} + \frac{1}{4} \geq x_{2B} + x_{3B}$.

Can we make Division Envy-Free?

Inequalities for Envy Free:

Alice not envious of Bob: $x_{2A} + x_{3A} + x_{4A} + \frac{1}{4} \geq x_{2B} + x_{3B}$.

Alice not envious of Carol: $x_{2A} + x_{3A} + x_{4A} + \frac{1}{4} \geq x_{3C} + x_{4C}$.

Can we make Division Envy-Free?

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All Constraints for Envy Free

All Constraints for Envy Free

$$\begin{aligned}
x_{2A} + x_{2B} &= \frac{1}{8} \\
x_{3A} + x_{3B} + x_{3C} &= \frac{5}{12} \\
x_{4A} + x_{4C} &= \frac{1}{12} \\
x_{2A} + x_{3A} + x_{4A} + \frac{1}{4} &\geq x_{2B} + x_{3B} \\
x_{2A} + x_{3A} + x_{4A} + \frac{1}{4} &\geq x_{3C} + x_{4C} \\
\frac{1}{8} + x_{2B} + x_{3B} &\geq x_{2A} + x_{3A} \\
\frac{1}{8} + x_{2B} + x_{3B} &\geq x_{3C} \\
x_{3C} + x_{4C} &\geq x_{3A} + x_{4A} \\
x_{3C} + x_{4C} &\geq x_{3B}
\end{aligned}$$

Final Reckoning- Envy Free

Final Reckoning- Envy Free

Maximize Total:

Final Reckoning- Envy Free

Maximize Total:

Alice: $x_{1A} = 0$, $x_{2A} = 0$, $x_{3A} = 0.1111$, $x_{4A} = 0$, $x_{5A} = 0.25$.

Final Reckoning- Envy Free

Maximize Total:

Alice: $x_{1A} = 0$, $x_{2A} = 0$, $x_{3A} = 0.1111$, $x_{4A} = 0$, $x_{5A} = 0.25$.

$$\frac{8}{7}(0 + 0 + 0.1111 + +0 + 0 + 0.25) \sim 0.4126$$

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Bob: $x_{1B} = 0.125$, $x_{2B} = 0.125$, $x_{3B} = 0.02777$, $x_{4B} = 0$, $x_{5B} = 0$.

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$$\frac{3}{2}(0.125 + 0.125 + 0.02778 + 0 + 0) \sim 0.41667$$

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Carol: $x_{1C} = 0$, $x_{2C} = 0$, $x_{3C} = 0.2777$, $x_{4C} = 0.08333$, $x_{5C} = 0$.

$$2(0 + 0 + 0.2777 + 0.08333) \sim 0.722$$

Final Reckoning- Envy Free

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TOTAL:

Final Reckoning- Envy Free

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TOTAL:

$$0.4162 + 0.4166 + 0.722 = 1.5512$$

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$$2(0 + 0 + 0.2777 + 0.08333) \sim 0.722$$

TOTAL:

$$0.4162 + 0.4166 + 0.722 = 1.5512$$

MOST UNHAPPY: Alice with 0.4126.

Minimize Unhappiness

Minimize Unhappiness

Got same numbers as wanted just proportional and min unhappiness.

Protocol

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Envy Free Protocol for n players, all have uniform valuations.

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Protocol

Envy Free Protocol for n players, all have uniform valuations.

1) Every player simul reveals their valuation. (honestly)

2) Players form LP program to satisfy that there is no envy, all vars make sense, and total is maximized. (They set the obv vars to 0 and whatever else is forced.) They solve the LP.

Player make the cuts as the LP solution dictates.

- How many cuts? As before $\le 2n^2 - n - 2$.
- Does this LP always have a solution? Yes.
- The paper ENDS has an $O(n^2)$ protocol for envy-free (hence prop) but does not maximize total. Extends to piece-wise valuations but with diff bound depending on number-of-pieces.

Other Valuations

Other Valuations

What if Valuation is of
$v(c,d) = \int_c^d (ax+b)dx = \frac{a}{2}(d^2 - c^2) + b(d-c)$.

Only makes sense if $1 = v(0,1) = \int_0^1 (ax+b)dx = \frac{a}{2} + b$.

$1 = \frac{a}{2} + b$

We do an example.

Example

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Let $f(x) = 2x$, $g(x) = x + \frac{1}{2}$, $h(x) = \frac{x}{2} + \frac{3}{4}$.

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Alice's Val: $val_A(b, a) = \int_a^b f(x) = b^2 - a^2$.

Example

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Alice's Val: $val_A(b,a) = \int_a^b f(x) = b^2 - a^2$.

Bob's Val: $val_B(b,a) = \int_a^b g(x) = \frac{1}{2}(b^2 - a^2) + \frac{1}{2}(b - a)$.

Example

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Alice's Val: $val_A(b, a) = \int_a^b f(x) = b^2 - a^2$.

Bob's Val: $val_B(b, a) = \int_a^b g(x) = \frac{1}{2}(b^2 - a^2) + \frac{1}{2}(b - a)$.

Carol's Val: $val_C(b, a) = \int_a^b h(x) = \frac{1}{4}(b^2 - a^2) + \frac{3}{4}(b - a)$.

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Note: $f(x), g(x), h(x)$ all MEET at $(\frac{1}{2}, 1)$.

Intervals

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This is DIFF than before.

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$$\begin{array}{ccccccccccccc} 0 & -- & x_1 & -- & x_2 & -- & \frac{1}{2} & -- & x_3 & -- & x_4 & -- & 1 \\ & C & & B & & A & & A & & B & & C & \end{array}$$

Intervals

This is DIFF than before.

$$0 \underset{C}{--} \quad x_1 \quad \underset{B}{--} \quad x_2 \quad \underset{A}{--} \quad \frac{1}{2} \quad \underset{A}{--} \quad x_3 \quad \underset{B}{--} \quad x_4 \quad \underset{C}{--} \quad 1$$

- A gets $[x_2, \frac{1}{2}] \cup [\frac{1}{2}, x_3]$
- B gets $[x_1, x_2] \cup [x_3, x_4]$
- C gets $[0, x_1] \cup [x_4, 1]$

Who Gets What?

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$$0 -- \quad x_1 \quad -- \quad x_2 \quad -- \quad \tfrac{1}{2} \quad -- \quad x_3 \quad -- \quad x_4 \quad -- \quad 1$$

$$C \quad B \quad A \quad A \quad B \quad C$$

Who Gets What?

$$0 \text{ --- } x_1 \text{ --- } x_2 \text{ --- } \tfrac{1}{2} \text{ --- } x_3 \text{ --- } x_4 \text{ --- } 1$$

$$\quad C \qquad B \qquad A \qquad A \qquad B \qquad C$$

A gets

$$\left(\frac{1}{2}\right)^2 - x_2^2 + x_3^2 - \left(\frac{1}{2}\right)^2 = x_3^2 - x_2^2$$

Who Gets What?

$$0 -- \quad x_1 \quad -- \quad x_2 \quad -- \quad \tfrac{1}{2} \quad -- \quad x_3 \quad -- \quad x_4 \quad -- \quad 1$$
$$C \qquad B \qquad A \qquad A \qquad B \qquad C$$

A gets

$$\left(\frac{1}{2}\right)^2 - x_2^2 + x_3^2 - \left(\frac{1}{2}\right)^2 = x_3^2 - x_2^2$$

B gets

$$\frac{1}{2}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{2}(x_2 - x_1 + x_4 - x_3)$$

Who Gets What?

$$\begin{array}{ccccccccccccc} 0 -- & x_1 & -- & x_2 & -- & \frac{1}{2} & -- & x_3 & -- & x_4 & -- & 1 \\ C & & B & & A & & A & & B & & C & \end{array}$$

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B gets

$$\frac{1}{2}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{2}(x_2 - x_1 + x_4 - x_3)$$

C gets

$$\frac{1}{4}(x_1^2 + 1 - x_4^2) + \frac{3}{4}(x_1 + 1 - x_4)$$

Alice's View of the World

Alice thinks:

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Alice gets $x_3^2 - x_2^2$

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Alice's View of the World

Alice thinks:

Alice gets $x_3^2 - x_2^2$

Bob gets $x_2^2 - x_1^2 + x_4^2 - x_3^2$

Carol gets $x_1^2 + 1 - x_4^2$.

Alice's View of the World

Alice thinks:

Alice gets $x_3^2 - x_2^2$

Bob gets $x_2^2 - x_1^2 + x_4^2 - x_3^2$

Carol gets $x_1^2 + 1 - x_4^2$.

Equations so that Alice has no envy:

Alice's View of the World

Alice thinks:

Alice gets $x_3^2 - x_2^2$

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Carol gets $x_1^2 + 1 - x_4^2$.

Equations so that Alice has no envy:

$x_3^2 - x_2^2 \geq x_2^2 - x_1^2 + x_4^2 - x_3^2$

Alice's View of the World

Alice thinks:

Alice gets $x_3^2 - x_2^2$

Bob gets $x_2^2 - x_1^2 + x_4^2 - x_3^2$

Carol gets $x_1^2 + 1 - x_4^2$.

Equations so that Alice has no envy:

$x_3^2 - x_2^2 \geq x_2^2 - x_1^2 + x_4^2 - x_3^2$

$x_3^2 - x_2^2 \geq x_1^2 + 1 - x_4^2$.

Bob's View of the World

Bob's View of the World

Bob thinks:

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Bob thinks:

Alice gets $\frac{1}{2}(x_3^2 - x_2^2) + \frac{1}{2}(x_3 - x_2)$

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Bob thinks:

Alice gets $\frac{1}{2}(x_3^2 - x_2^2) + \frac{1}{2}(x_3 - x_2)$

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Carl gets $\frac{1}{2}(x_1^2 + 1 - x_4^2) + \frac{1}{2}(x_1 + 1 - x_4)$

Bob's View of the World

Bob thinks:

Alice gets $\frac{1}{2}(x_3^2 - x_2^2) + \frac{1}{2}(x_3 - x_2)$

Bob gets $\frac{1}{2}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{2}(x_2 - x_1 + x_4 - x_3)$

Carl gets $\frac{1}{2}(x_1^2 + 1 - x_4^2) + \frac{1}{2}(x_1 + 1 - x_4)$

Equations so that Bob has no envy:

Bob's View of the World

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Alice gets $\frac{1}{2}(x_3^2 - x_2^2) + \frac{1}{2}(x_3 - x_2)$

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Carl gets $\frac{1}{2}(x_1^2 + 1 - x_4^2) + \frac{1}{2}(x_1 + 1 - x_4)$

Equations so that Bob has no envy:

$(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_3^2 - x_2^2) + (x_3 - x_2)$

Bob's View of the World

Bob thinks:

Alice gets $\frac{1}{2}(x_3^2 - x_2^2) + \frac{1}{2}(x_3 - x_2)$

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Equations so that Bob has no envy:

$$(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_3^2 - x_2^2) + (x_3 - x_2)$$

$$(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4)$$

Carol's View of the World

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Carol thinks:

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Carol's View of the World

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Alice gets $\frac{3}{4}(x_3^2 - x_2^2) + \frac{1}{4}(x_3 - x_2)$

Bob gets $\frac{3}{4}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{4}(x_2 - x_1 + x_4 - x_3)$

Carol's View of the World

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Alice gets $\frac{3}{4}(x_3^2 - x_2^2) + \frac{1}{4}(x_3 - x_2)$

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Carol gets $\frac{3}{4}(x_1^2 + 1 - x_4^2) + \frac{1}{4}(x_1 + 1 - x_4)$

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Alice gets $\frac{3}{4}(x_3^2 - x_2^2) + \frac{1}{4}(x_3 - x_2)$

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Carol gets $\frac{3}{4}(x_1^2 + 1 - x_4^2) + \frac{1}{4}(x_1 + 1 - x_4)$

Equations so that Bob has no envy:

Carol's View of the World

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Alice gets $\frac{3}{4}(x_3^2 - x_2^2) + \frac{1}{4}(x_3 - x_2)$

Bob gets $\frac{3}{4}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{4}(x_2 - x_1 + x_4 - x_3)$

Carol gets $\frac{3}{4}(x_1^2 + 1 - x_4^2) + \frac{1}{4}(x_1 + 1 - x_4)$

Equations so that Bob has no envy:

$3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_3^2 - x_2^2) + (x_3 - x_2)$

Carol's View of the World

Carol thinks:

Alice gets $\frac{3}{4}(x_3^2 - x_2^2) + \frac{1}{4}(x_3 - x_2)$

Bob gets $\frac{3}{4}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{4}(x_2 - x_1 + x_4 - x_3)$

Carol gets $\frac{3}{4}(x_1^2 + 1 - x_4^2) + \frac{1}{4}(x_1 + 1 - x_4)$

Equations so that Bob has no envy:

$$3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_3^2 - x_2^2) + (x_3 - x_2)$$

$$3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3)$$

Problem 1:

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$$x_3^2 - x_2^2 \geq x_1^2 + 1 - x_4^2.$$

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$x_3^2 - x_2^2 \geq x_2^2 - x_1^2 + x_4^2 - x_3^2$
$x_3^2 - x_2^2 \geq x_1^2 + 1 - x_4^2.$

$(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_3^2 - x_2^2) + (x_3 - x_2)$
$(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4)$

Problem 1:

Problem 1: Does there exist x_1, x_2, x_3, x_4 that satisfies the following inequalities:

$0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1$

$$x_3^2 - x_2^2 \geq x_2^2 - x_1^2 + x_4^2 - x_3^2$$
$$x_3^2 - x_2^2 \geq x_1^2 + 1 - x_4^2.$$

$$(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_3^2 - x_2^2) + (x_3 - x_2)$$
$$(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4)$$

$$3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_3^2 - x_2^2) + (x_3 - x_2)$$
$$3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3)$$

Note: Can Phrase as Quad Prog Problem.

Quadratic Programming

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The Quadratic Programming Problem Maximize (or Minimize) a LINEAR function relative to QUADRATIC constraints.

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Relative to

$$\begin{aligned}
-3x^2 + 5y - 8z^2 &\le 20 \\
x^2 + y^2 + z &\le 5 \\
2x + y^2 + 18z &\le 100 \\
7x + 29y + 178z^2 &\le 193
\end{aligned}$$

- NP-Hard. Thought to be HARD.
- There is ONE PACKAGES for it that I know.

Problem 2:

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Problem 2: Maximize

$$\left(\frac{1}{2}\right)^2 - x_2^2 + x_3^2 - \left(\frac{1}{2}\right)^2 + x_3^2 - x_2^2 + \frac{1}{2}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{2}(x_2 - x_1 + x_4 - x_3)$$

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while satisfying:

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FML!!! My prof wants me to solve a QQP!!!

Protocol

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Envy Free Protocol for n players, all have linear valuations.

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4) If there are ≥ 2 people left when solved then use the solution. If there is only 1 person left, he gets it.

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3) Cut the cake as it dictates.

- Does a QQP of his form always have a solution?
- Is there always a rational point that satisfies the constraints? Unlikely.
- Is there an efficient algorithm to find an approx solution to the QQP that arise from this problem? (Do not know?)
- Will these be solved before or after the Gov. Shutdown ends?