Discrete Envy Free Protocols

William Gasarch-U of MD

A 3-Player Discrete Envy Free Protocol

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$n = 3$ Phase I

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B not envious since she went first.

4) C takes a piece thats left but must take P_3' if its left.

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C not envious since he created a tie for first.

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5) A takes the piece thats left.

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A not envious since he gets untrimmed piece.

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3) B takes one of P_1, P_2, P_3'. (biggest)
B not envious since she went first.

4) C takes a piece thats left but must take P_3' if its left.
C not envious since he created a tie for first.

5) A takes the piece thats left.

A not envious since he gets untrimmed piece.

Good News: the allocation of P_1, P_2, P_3' is envy-free

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3) B takes one of P_1, P_2, P_3'. (biggest)
B not envious since she went first.

4) C takes a piece thats left but must take P_3' if its left.
C not envious since he created a tie for first.

5) A takes the piece thats left.

A not envious since he gets untrimmed piece.

Good News: the allocation of P_1, P_2, P_3' is envy-free

Bad News: What to do about T?

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Key A thinks $P_1 = P_3' + T$. If C gets **all** of T, A doesn't care.

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Terminology A has **an advantage over C** with regard to T.
In Phase 2 T is divided.

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1) B cuts T into 3 pieces (equal)

2) C takes a piece. A takes a piece. B takes a piece.

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C not envious since she went first.

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C not envious since she went first.

B not envious since he thinks all the pieces are equal.

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In Phase 2 T is divided.

1) B cuts T into 3 pieces (equal)

2) C takes a piece. A takes a piece. B takes a piece.

C not envious since she went first.

B not envious since he thinks all the pieces are equal.

A not envious of B since A picked before B.

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Recap We will assume

A got P_1.

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Key A thinks $P_1 = P_3' + T$. If C gets **all** of T, A doesn't care.

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2) C takes a piece. A takes a piece. B takes a piece.

C not envious since she went first.

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C not envious since she went first.
B not envious since he thinks all the pieces are equal.
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That last case was the interesting case.

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Terminology A has **an advantage over C** with regard to T.
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That last case was the interesting case.

Since P_1, P_2, P_3' are split envy-free, and T is split envy-free, we have an envy-free division.

An all-but-ϵ Envy-Free Protocol for 4 People

William Gasarch-U of MD

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We will show:
For every $\epsilon > 0$ there is an all-but-ϵ envy-free protocol for 4 people.

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We will show:
For every $\epsilon > 0$ there is an all-but-ϵ envy-free protocol for 4 people.

We will present some sub-protocols and then combine them.

The First-Player-Happy Protocol

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A, B, C, D are the players.

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1) All but T of the pie is allocated.

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3) The allocation is envy-free.

4) We do not know what B, C, D think of T.

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5) C takes a piece. (Must take piece she trimmed if available)
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6) B takes a piece. (Must take a piece he trimmed if available)
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7) A takes a piece. She gets untrimmed piece so no envy

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There is Trim and a piece left over.

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7) A takes a piece. She gets untrimmed piece so no envy
There is Trim and a piece left over. Envy-free on what was divided.
Needed to divide into 5 pieces since if D took untrimmed A would get trimmed.

The First-Player-Happy Protocol (cont)

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Claim 1 A gets $1/5$. Hence A thinks that there is $4/5$ left.

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We look at all the players.

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A A splits P into 5 equal pieces.

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≤ 3 pieces trimmed, so ≥ 2 pieces untrimmed.

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A gets untrimmed piece so gets $\frac{1}{5}$. She's happy!

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We look at all the players.

A A splits P into 5 equal pieces.

≤ 3 pieces trimmed, so ≥ 2 pieces untrimmed.

A gets untrimmed piece so gets $\frac{1}{5}$. She's happy!

We do not know what B, C, D think about their pieces accept that the allocation is envy-free.

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1) FIRST-PLAYER-HAPPY($A, B, C, D; P$).

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The trim and piece set aside is S_2.

The All-Players-Happy Protocol

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3) FIRST-PLAYER-HAPPY$(C, D, A, B; S_2)$.

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3) FIRST-PLAYER-HAPPY$(C, D, A, B; S_2)$.
The trim and piece set aside is S_3.

The All-Players-Happy Protocol

1) FIRST-PLAYER-HAPPY$(A, B, C, D; P)$.
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3) FIRST-PLAYER-HAPPY$(C, D, A, B; S_2)$.
The trim and piece set aside is S_3.

4) FIRST-PLAYER-HAPPY$(D, A, B, C; S_3)$.

The All-Players-Happy Protocol

1) FIRST-PLAYER-HAPPY$(A, B, C, D; P)$.
The trim and piece set aside is S_1.

2) FIRST-PLAYER-HAPPY$(B, C, D, A; S_1)$.
The trim and piece set aside is S_2.

3) FIRST-PLAYER-HAPPY$(C, D, A, B; S_2)$.
The trim and piece set aside is S_3.

4) FIRST-PLAYER-HAPPY$(D, A, B, C; S_3)$.

Can show all players think they got $\geq \frac{1}{5}$ so $\leq \frac{4}{5}$ left.

All-but ϵ Envy-Free

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For all ϵ

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1) All but S of the pie is allocated.
2) The allocation is Envy-Free
3) All of the players think $S \leq \epsilon$.

All-but ϵ Envy-Free (cont)

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$S_0 = P$.

All-but ϵ Envy-Free (cont)

$S_0 = P$.
For $i = 0$ to N
    ALL-PLAYER-HAPPY$(A, B, C, D; S_i)$.
    Let S_{i+1} be all that was set aside.

All-but ϵ Envy-Free (cont)

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For $i = 0$ to N
    ALL-PLAYER-HAPPY$(A, B, C, D; S_i)$.
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After N iterations all of the players think $S_N \leq (\frac{4}{5})^N$.

All-but ϵ Envy-Free (cont)

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    ALL-PLAYER-HAPPY$(A, B, C, D; S_i)$.
    Let S_{i+1} be all that was set aside.

After N iterations all of the players think $S_N \leq (\frac{4}{5})^N$.

For N large enough get $S_N \leq \epsilon$.

Is there A Discrete Envy-Free Protocol for 4 People?

William Gasarch-U of MD

Can We Get Rid Of The ϵ?

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We just proved the following:

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Thm For all $\epsilon > 0$ there is a 4-player discrete envy-free protocol.

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Vote for one of the following:

The statement is known to be true.

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Vote for one of the following:

The statement is known to be true.

The statement is known to be false.

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We just proved the following:

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Consider the following statement:

There is a 4-player discrete envy-free protocol

Vote for one of the following:

The statement is known to be true.

The statement is known to be false.

The statement's truth is UNKNOWN TO SCIENCE!

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Consider the following statement:

There is a 4-player discrete envy-free protocol

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Answer on the next slide.

The Statement is True

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Thm There is a 4-player discrete envy-free protocol.

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We are not going to prove this.

The Statement is True

Thm There is a 4-player discrete envy-free protocol.

We are not going to prove this.

I will give you the main ideas behind the proof; however, the full proof has more going on than I can tell you in the time allocated.

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The answer is on the next page.

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Bill There are valuations V_A, V_B, V_C, V_D such that, if those are the valuations of A, B, C, D, then the protocol will take at ≥ 367 cuts.

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Bill I can and I will!

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A thinks $P < Q'$ since she thought $P = Q$ and she gave P a nonzero part of Q.

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Vote YES or NO.

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4) You can use Google Translate or just ask ChatGPT to tell you the protocol!