Unequal Division

William Gasarch-U of MD

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Discuss how they can do this.

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Should number-of-clones even be a complexity measure?
Ethical question: When you fuse them all into A (or B) is that murder?

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CON Number of cuts is 12. Surely we can do better than that.
Response Indeed we can, and don't call me Shirley.

There are Better Protocols But

There are better protocols but they are better on blackboard
So for now these slides are at an end.

Protocols Bill Does on Blackboard

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(8:5) with 5 cuts

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$(a : b)$ - see next slide for two approaches.

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For $(a : b)$ with $a < b$.

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If not then A takes it.

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Answer on next slide

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The paper is here: `https://arxiv.org/abs/1206.1553`