

Intermediary Truth Values

TRUE is 1, FALSE is 0

We have dealt with **math** where statement really are **true** or **false** (with some rare exceptions).

The **Real World** is messier!

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- ▶ Its going to rain tomorrow. Forecasts give probabilities.

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We will **not** dwell on this. We will ponder a well defined math question about intermediary truth values.

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2. Want the definitions to satisfy De Morgan' Law.
3. Want the definitions to make sense intuitively. For example,
 $x \wedge y \leq x$ (harder for $x \wedge y$ to be true then for x to be true)
 $x \vee y \geq x$ (easier for $x \vee y$ to be true then for x to be true)

Work on in groups!

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Then let $x \wedge y = f(x, y)$ and $x \vee y = 1 - f(1 - x, 1 - y)$.

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