0.1 Mono Image Solutions and Mono Solutions

We need to consider the following variant on mono solutions.

Definition 0.1 Let $p_1(z_1, z_2)$, ..., $p_m(z_1.z_2) \in \mathsf{Z}[z_1, z_2]$. We denote this set of polynomials by F.

1. An *image-solution of* F is just $(a, d; y_1, \ldots, y_m)$ where $(\forall 1 \le i \le m)[y_i = p_i(a, d)]$.
2. Assume the reals are 2-colored. Then a *mono image-solution of* F is an image-solution $(a, d; y_1, \ldots, y_m)$ wher $y_1, \ldots, y_m$ are all the same color. Note that a, d need not be that color.
3. A *Red image-solution of* F and a *Blue image-solution of* F have the obvious meaning.

Lemma 0.2 *Let* $m \in \mathsf{N}$*. We assume* m *is a cube. Let* G *be the system of equations*

$$(\forall 2 \le i \le m-1)[y_{i+1} = 2y_i - y_{i-1} + 2].$$

There exists a set of polynomials F*:*

$$p_1(z_1, z_2), p_2(z_1, z_2), \ldots, p_m(z_1, z_2) \in \mathsf{Z}[z_1, z_2]$$

such that the following holds:

C1) For all i*,* $p_i(z_1, z_2)$ *is linear in* z_1, z_2*.*

C2) The coefficients of $p_i(z_1, z_2)$ *are quadratic polynomials in* i *over* Z*.*

C3) If $a, d \in [0, m^{1/3}]$ *then, for all* i*,* $0 \le p_i(a, d) \le 2m^2$

C4) (a) If $(a, d; y_1, \ldots, y_m)$ *is an image-solution of* F *then* $(y_1, \ldots, y_m)$ *is a solution to* G*.*

(b) If $(y_1, \ldots, y_m)$ *is a solution to* G *then* $(a, d; y_1, \ldots, y_m)$*, where* $a = y_1$ *and* $d = y_2 - y_1$*, is an image-solution of* F*.*

Proof:

We motivate the definition of $\{p_i(z_1, z_2)\}_{i=1}^m$ by assuming that G has a solution $(y_1, \ldots, y_m)$. Let

$y_1 = a$
$y_2 = a + d.$
Then

$y_3 = 2y_2 - y_1 + 2 = 2(a + d) - a + 2 = a + 2d + 2$
$y_4 = 2y_3 - y_2 + 2 = 2a + 4d + 4 - (a + d) + 2 = a + 3d + 6$

One can show by induction that

$(\forall 1 \le i \le m)[y_i = a + (i-1)d + (i^2 - 3i + 2)].$

We define p_i as follows. For $1 \le i \le m$ let

$$p_i(z_1, z_2) = z_1 + (i-1)z_2 + (i^2 - 3i + 2).$$

Properties 1,2,3 for F are easy to verify. We show property 4 which is essentially true by construction.

If G has a solution $(y_1, \ldots, y_m)$ then let $y_1 = a$ and $y_2 = a + d$. Then F has image-solution $(a, d; y_1, \ldots, y_m)$

If F has a solution $(a, d; y_1, \ldots, y_m)$ then G has solution

$y_1 = a$

$y_2 = a + d$

$(\forall 3 \le i \le m)[y_i = a + (i-1)d + (i^2 - 3i + 2) = p_i(a, d)]$. ▮

0.2 Colorings Where There is no Blue Mono Image-Solution to ...

Lemma 0.3 *Let q be a prime and let $m \ge q^3$. Let $p_1(z_1, z_2), \ldots, p_m(z_1, z_2) \in \mathsf{Z}[z_1, z_2]$ be such that the following hold:*

C1) For all i, $p_i(z_1, z_2)$ is linear in z_1, z_2.

C2) The coefficients of $p_i(z_1, z_2)$ are quadratic polynomials in i over Z.

C3) If $a, d \in [0, q]$ then, for all i, $0 \le p_i(a, d) \le 2m^2$

Let b, r be such that $0 \le b, r \le 1$ and $b + r = 1$. Let COL be as in Definition **??**.
THEN
the probability there is a blue image-solution to $p_1, \ldots, p_m$ is bounded above by $2500m^6 b^{m/6}$.

Proof:

The proof is in two parts

PART ONE: The Set of Intervals Mod q.

Fix $a, d \in \mathsf{R}$. We want to bound

$$\Pr(\mathrm{COL}(p_1(a, d)) = \cdots = \mathrm{COL}(p_m(a, d)) = \mathrm{BLUE}).$$

Recall that $\mathrm{COL}(z) = \mathrm{COL}'(z \bmod q)$. Hence, in order to have

$$\mathrm{COL}(p_1(a, d)) = \cdots = \mathrm{COL}(p_m(a, d)) = \mathrm{BLUE})$$

we need to have the following happen:

- $p_1(a, d) \bmod q$ is in an interval that COL$'$ colors BLUE. This occurs with probability b.
- $p_2(a, d) \bmod q$ is in an interval that COL$'$ colors BLUE. This occurs with probability b.
- Etc until $p_m(a, d) \bmod q$ is in an interval that COL$'$ colors BLUE. This occurs with probability b.

At first glance you might think the probability of this happening is b^m which is small, so good news for us. Alas no. Here is an extreme possibility (that we later show cannot happen): all of the $p_i(a,d) \bmod q$ are in the same interval. This raises the question: how many distinct intervals do we get?

Let $F(x) = p_x(a,d)$. By premise 2, $F(x)$ is quadratic. Also recall that q is prime and $m \ge q^3$. Hence $F(x), m, q$ satisfy the premise of Theorem **??**. Therefore

$$\{F(1) \bmod q, \ldots, F(m) \bmod q\}$$

will intersect $\ge q/6$ intervals. Hence

$$\{p_1(a,d) \bmod q, \ldots, p_m(a,d) \bmod q\}$$

will intersect $\ge q/6$ intervals.

Hence

$$\Pr(\mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE}) \le b^{q/6}.$$

PART TWO: How Many Sets of Intervals?

The statement

$$\Pr(\mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE}) \le b^{q/6}.$$

is not about a, d: its about interval patterns. That is, we map a, d to the set of intervals mod q that

$$p_1(a,d) \bmod q, \ldots, p_m(a,d) \bmod q$$

are in and then we look at the probability that all of those intervals are the same color.

So the question is: how many sets of intervals can there be?

Since the coloring is mod q we can assume $a, d \in [0, q)$. By premise 3, $0 \le p_i(a,d) \le 2m^2$. We now ask about the intervals (not mod q).

Consider the following questions:

- Of the intervals $[0,1)$, $[1,2)$, $\ldots$, $[2m^2-1, 2m^2)$ which one has $p_1(a,d)$? There are $2m^2$ possibilities. Note that which interval can be determined from the sign changes of the following sequence:

  $$p_1(a,d) - 1, p_1(a,d) - 2), \ldots, p_1(a,d) - 2m^2.$$

- Of the intervals $[0,1)$, $[1,2)$, $\ldots$, $[2m^2-1, 2m^2)$ which one has $p_2(a,d)$? There are $2m^2$ possibilities.

  Note that which interval can be determined from the sign changes of the following sequence:

  $$p_2(a,d) - 1, p_2(a,d) - 2), \ldots, p_2(a,d) - 2m^2.$$

- Etc until Of the intervals $[0,1)$, $[1,2)$, $\ldots$, $[2m^2-1, 2m^2)$ which one has $p_m(a,d)$? There are $2m^2$ possibilities.

  Note that which interval can be determined from the sign changes of the following sequence:

  $$p_m(a,d) - 1, p_m(a,d) - 2), \ldots, p_m(a,d) - 2m^2.$$

At a first glance it would seem like there are $(2m^2)^m$ possibilities. There are far less.

Consider the sequence of polynomials

$$p_1(z_1, z_2) - 1, \ldots, p_1(z_1, z_2) - 2m^2,$$

$$p_2(z_1, z_2) - 1, \ldots, p_2(z_1, z_2) - 2m^2,$$

$$\vdots$$

$$p_m(z_1, z_2) - 1, \ldots, p_m(z_1, z_2) - 2m^2.$$

The maximum number of sign changes in this sequence is an upper bound on the number of ways $p_1(a, d)$, ..., $p_m(a, d)$ can be in the intervals.

By Lemma **??**.2 the number of sign changes is $\le 625(2m^3)^2 = 2500m^6$. Hence the number of ways $p_1(a, d), \ldots, p_m(a, d)$ are in the intervals is $\le 2500m^6$.

COMBINE PARTS ONE AND TWO

Map $(a, d) \in \mathsf{R}^2$ to the set of $\ge q/6$ intervals that contain

$$p_1(a, d) \bmod q, \ldots, p_m(a, d) \bmod q.$$

There are at most $2500m^6$ sets of intervals. Each one has probability $\le b^{q/6}$ of having them all be blue. By the union bound the probability that one of those sets has every interval blue is

$$\le 2500m^6 b^{q/6}.$$

∎

0.3 Colorings Where There is no Blue Solution to ...

By combining Lemmas 0.2 and 0.3 we obtain the following.

Theorem 0.4 *Let $m \in \mathsf{N}$. We assume m is a cube. Let G be the system of equations*

$$(\forall 2 \le i \le m-1)[y_{i+1} = 2y_i - y_{i-1} + 2].$$

Let b, r be such that $0 \le b, r \le 1$ and $b + r = 1$. Let COL *be as in Definition* **??**.

THEN

the probability there is a blue solution to G is bounded above by $2500m^6b^{m/6}$.