

Sum of the Inverse Catalan Numbers

by William Gasarch

This is not original to me, but I don't know where I got it from. Possibly chatgpt.

Def 0.1 $C_n = \frac{1}{n+1} \binom{2n}{n}$.

We show

$$\sum_{n=0}^{\infty} \frac{1}{C_n} = 2 + \frac{4\pi\sqrt{3}}{27} \approx 2.806133050770763489.$$

1 Needed Lemmas On Integrals

Lemma 1.1

1. For all $a, b \geq 1$,

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{(a-1)!(b-1)!}{(a+b-1)!}$$

2.

$$\int_0^1 t^{n-1}(1-t)^n dt = \frac{(n-1)!n!}{(2n)!}$$

(This follows from Part 1.)

Proof:

We prove this by induction on b .

Base Case: Assume $a \geq 1$ and $b = 1$.

$$\int_0^1 t^{a-1} dt = \frac{(a-1)!(1-1)!}{(a+1-1)!} = \frac{(a-1)!}{a!} = \frac{1}{a}.$$

This integral is easily seen to be true.

IH (The $b-1$ case). For all $a \geq 1$, $\int_0^1 t^{a-1}(1-t)^{b-2} dt = \frac{(a-1)!(b-2)!}{(a+b-2)!}$.

We will need this in the case of $a+1$ so we state that here:

$$\int_0^1 t^a(1-t)^{b-2} dt = \frac{a!(b-2)!}{(a+b-1)!}$$

IS We can assume $b \geq 2$.

We look at:

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt$$

We do integration by parts.

$$u = (1-t)^{b-1} \text{ so } du = -(b-1)(1-t)^{-2} dt.$$

$$dv = t^{a-1} dt \text{ so } v = \frac{t^a}{a}.$$

Note that uv is 0 at both $t = 0$ and $t = 1$ so $uv \Big|_0^1 = 0$.

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt = - \int_0^1 -\frac{1}{a} t^a (b-1)(1-t)^{b-2} dt = \frac{b-1}{a} \int_0^1 t^a (1-t)^{b-2} dt.$$

By the IH we know

$$\int_0^1 t^a (1-t)^{b-2} dt = \frac{a!(b-2)!}{(a+b-1)!}$$

Hence we have

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{b-1}{a} \frac{a!(b-2)!}{(a+b-1)!} = \frac{(a-1)!(b-1)!}{(a+b-1)!}.$$

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2 Needed Lemmas on Sums

Lemma 2.1

1. $\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}.$
2. $\sum_{i=1}^{\infty} i^2x^i = \frac{x(1+x)}{(1-x)^3}.$
3. $\sum_{i=1}^{\infty} i(i+1)x^i = \frac{2x}{(1-x)^3}.$

Proof:

1)

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

Take the derivative of both sides.

$$\sum_{i=0}^{\infty} ix^{i-1} = \frac{1}{(1-x)^2}$$

The first term of the sum is 0 so it can be omitted. Also we multiply both sides by x .

$$\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$$

2)

$$\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$$

Take the derivative of both sides.

$$\sum_{i=1}^{\infty} i^2x^{i-1} = \frac{1+x}{(1-x)^3}$$

Multiply both sides by x

$$\sum_{i=1}^{\infty} i^2x^i = \frac{x(1+x)}{(1-x)^3}$$

3)

$$\begin{aligned}\sum_{i=1}^{\infty} i(i+1)x^i &= \sum_{i=1}^{\infty} i^2 x^i + \sum_{i=1}^{\infty} i x^i \\ &= \frac{x}{(1-x)^2} + \frac{x(1+x)}{(1-x)^3} = \frac{x(1-x)}{(1-x)^3} + \frac{x(1+x)}{(1-x)^3} = \frac{2x}{(1-x)^3}.\end{aligned}$$

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Lemma 2.2

1. $1 + \tan^2 \theta = \sec^2 \theta$
2. $\frac{1}{(1+\tan^2 \theta)^3} = \cos^6 \theta$ (*This follows from Part 1.*)

Proof:

$$\sin^2 \theta + \cos^2 \theta = 1$$

Divide by $\cos^2 \theta$.

$$\frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \sec^2 \theta$$

$$\tan^2 \theta + 1 = \sec^2 \theta. \quad \blacksquare$$

3 Needed Lemma on $\int \cos^n x dx$

Let

$$I_n = \int \cos^n x dx.$$

We derive a formula for I_n in terms of I_{n-2} .

Start with

$$I_n = \int \cos^{n-1} x \cos x dx.$$

Use integration by parts. Let

$$u = \cos^{n-1} x$$

and

$$dv = \cos x dx.$$

Then

$$du = -(n-1) \cos^{n-2} x \sin x dx$$

and

$$v = \sin x.$$

Therefore,

$$I_n = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \sin^2 x dx.$$

Use

$$\sin^2 x = 1 - \cos^2 x.$$

Then

$$\begin{aligned} I_n &= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx \\ &= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx. \end{aligned}$$

Since

$$I_n = \int \cos^n x dx$$

and

$$I_{n-2} = \int \cos^{n-2} x dx,$$

we have

$$I_n = \sin x \cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n.$$

Hence

$$nI_n = \sin x \cos^{n-1} x + (n-1)I_{n-2}.$$

Dividing by n ,

$$I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}.$$

Thus

$$\int \cos^n x \, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

EXAMPLE: $n = 4$

The reduction formula gives

$$\int \cos^4 x \, dx = \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \int \cos^2 x \, dx.$$

Also,

$$\cos^2 x = \frac{1 + \cos(2x)}{2},$$

so

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin(2x)}{4}.$$

Therefore,

$$\int \cos^4 x \, dx = \frac{1}{4} \sin x \cos^3 x + \frac{3x}{8} + \frac{3 \sin(2x)}{16} + C.$$

We will need this later:

Corollary 3.1 $\int_{-\pi/6}^{\pi/6} \cos^4 \theta \, d\theta = \frac{\pi}{8} + \frac{9\sqrt{3}}{32}$

4 Main Theorem

Theorem 4.1

$$\sum_{n=0}^{\infty} \frac{1}{C_n} = 2 + \frac{4\pi\sqrt{3}}{27} \approx 2.806133050770763489.$$

Proof:

For $n \geq 1$,

$$\frac{1}{C_n} = \frac{(n+1)(n!)^2}{(2n)!}.$$

By Lemma 1.1.2:

$$\int_0^1 t^{n-1}(1-t)^n dt = \frac{(n-1)!n!}{(2n)!}.$$

Hence

$$\frac{1}{C_n} = n(n+1) \int_0^1 t^{n-1}(1-t)^n dt.$$

so

$$\sum_{n=1}^{\infty} \frac{1}{C_n} = \int_0^1 \frac{1}{t} \sum_{n=1}^{\infty} n(n+1)(t(1-t))^n dt.$$

By Lemma 2.1.2:

$$\sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3},$$

Hence we get

$$\sum_{n=1}^{\infty} \frac{1}{C_n} = \int_0^1 \frac{2(1-t)}{(1-t+t^2)^3} dt$$

Let $t = x + B$ where we determine B later

$$(x+B)^2 - (x+B) + 1 = x^2 + 2Bx + B^2 - x - B + 1 = x^2 + (2B-1)x + (B^2 - B + 1)$$

We can choose $B = \frac{1}{2}$ so that coefficient of x is 0.

$$t = x + \frac{1}{2}.$$

$$dt = dx.$$

$$x = t - \frac{1}{2}.$$

So

Since

$$2(1-t) = 2(1-(x+\frac{1}{2})) = 2(1-x-\frac{1}{2}) = 2(\frac{1}{2}-x) = 1-2x$$

We get

$$\int_0^1 \frac{2(1-t)dt}{(t^2-t+1)^3} = \int_{-1/2}^{1/2} \frac{(1-2x)dx}{(x^2+3/4)^3} = \int_{-1/2}^{1/2} \frac{dx}{(x^2+3/4)^3} - \int_{-1/2}^{1/2} \frac{2x dx}{(x^2+3/4)^3}$$

Since $\frac{2x}{(x^2+3/4)^3}$ is an odd function,

$$\int_{-1/2}^{1/2} \frac{2x dx}{(x^2+3/4)^3} = 0$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{C_n} = \int_{-1/2}^{1/2} \frac{dx}{(x^2+3/4)^3}$$

We want x^2 to have coefficient $3/4$. We set

$$x = \frac{\sqrt{3}}{2}y.$$

$$dx = \frac{\sqrt{3}}{2}dy.$$

$$y = \frac{2x}{\sqrt{3}}.$$

$$\text{When } x = \frac{1}{2}, y = \frac{1}{\sqrt{3}}.$$

$$\text{When } x = -\frac{1}{2}, y = -\frac{1}{\sqrt{3}}.$$

So

$$\int_{-1/2}^{1/2} \frac{dx}{(x^2+3/4)^3} = \frac{\sqrt{3}}{2} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dy}{((3/4)y^2+3/4)^3} = \frac{32\sqrt{3}}{27} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dy}{(y^2+1)^3}$$

RECAP. We have:

$$\sum_{n=1}^{\infty} \frac{1}{C_n} = \frac{32\sqrt{3}}{27} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dx}{(1+x^2)^3}.$$

Letting $x = \tan \theta$ and using Lemma 2.2.2 we get:

$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dx}{(1+x^2)^3} = \int_{-\pi/6}^{\pi/6} \cos^4 \theta \, d\theta$$

By Corollary 3.1

$$\int_{-\pi/6}^{\pi/6} \cos^4 \theta \, d\theta = \frac{\pi}{8} + \frac{9\sqrt{3}}{32}.$$

Putting this all together we get

$$\sum_{n=1}^{\infty} \frac{1}{C_n} = 1 + \frac{4\pi\sqrt{3}}{27},$$

and since $C_0 = 1$,

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{C_n} = 2 + \frac{4\pi\sqrt{3}}{27}.}$$

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