# 1 PROJECT ONE: Random Ramsey Theory

1. General Pitch 1: We know that $R(3) = 6$. But if you color the edges of $K_5$ at random then the prob that you DO NOT get a mono $K_3$ is very small. What about other Ramsey Numbers?

2. General Pitch 2: We know that $R(3) = 6$. But if you color the edges of $K_6$ at random then HOW MANY mono $K_3$'s will you get? What about other Ramsey Numbers?

3. Write a program that does the following: Given $n, k$

   1) Randomly color the edges of $K_n$ and count the number of mono $K_k$'s there are.

   2) DO THAT 100 times (maybe less if this is to much for your computer).

4. Recall that $R(3) = 6$. That motivates the following experiment:

   a) If you 2-color the edges of $K_5$ 100 times Collect the data and see which number-of-$K_3$'s occurs the most times.

   b) If you 2-color the edges of $K_6$ 100 times Collect the data and see which number-of-$K_3$'s occurs the most times (it will be at least 1).

   c) If you 2-color the edges of $K_7$ 100 times Collect the data and see which number-of-$K_3$'s occurs the most times (it will be at least 1).

   $\vdots$

   Stop at $K_{10}$.

   Make conjectures about the following:

   i) If you randomly color the edges of $K_n$ then the expected number of mono $K_3$'s is XXX (for large $n$).

   ii) If you color the edges of $K_n$ then you will always get at least XXX mono $K_3$'s. (for large $n$).

5. Recall that $R(4) = 18$. That motivates the following experiment:

   a) If you 2-color the edges of $K_{10}$ 100 times Collect the data and see which number-of-$K_4$'s occurs the most times.

   b) If you 2-color the edges of $K_{11}$ 100 times Collect the data and see which number-of-$K_4$'s occurs the most times (it will be at least 1).

   c) If you 2-color the edges of $K_{12}$ 100 times Collect the data and see which number-of-$K_4$'s occurs the most times (it will be at least 1).

   $\vdots$

   Stop at $K_{25}$.

   Make conjectures about the following:

   i) Find the $m$ such that the following two hold:

   If you color the edges of $K_m$ at random then the prob of getting a mono $K_4$ is $\geq 0.9$.

   If you color the edges of $K_{m-1}$ at random then the prob of getting a mono $K_4$ is $< 0.9$.

ii) If you randomly color the edges of $K_n$ then the expected number of mono $K_4$'s is XXX (for large $n$).

iii) If you color the edges of $K_n$ then you will always get at least XXX mono $K_4$'s (for large $n$).

6. I will teach you other Ramsey Theorems that will lead to similar questions.

## 2 PROJECT TWO: Random VDW numbers

Same as PROJECT ONE except with VDW numbers. I give one example:

$W(3,2) = 9$.

For all COL[9] $\rightarrow$ [2] there exists a mono 3-AP.

We suspect that most COL[8] $\rightarrow$ [2] have a mono 3-AP (though there are some that don't).

Find the smallest $n$ such that, for all COL[$n$] $\rightarrow$ [2] the prob that there is a mono 3-AP is $\geq 0.9$.

## 3 PROJECT THREE: Ramsey Numbers via SAT Solvers

1. Investigate publicly available SAT solvers and what format they use. Try them out.

2. Phrase questions like

   *There is a 2-coloring of the edges of $K_{13}$ with no mono $K_4$ as a SAT question. Try the SAT solvers on them.*

3. Alternatively: Write your own SAT solvers and see how they do.

4. I have some other ideas about this topic as well.

## 4 PROJECT FOUR: Survey and Extend Cycle Results

Let $C_k$ be the cycle of length $k$.

Let $R(C_k)$ be the least number such that, for all 2-colorings of the edges of $K_n$, there is a mono $C_k$.

I think that all of the numbers $R(C_k)$ are known. This survey has the references needed:

`https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS1`

1. A survey of all that is known about $R(C_k)$. It might have to be about $R(C_{k_1}, C_{k_2})$.

2. $R(C_4) = 6$ is known. The proof that every COL: $\binom{[6]}{2} \rightarrow [2]$ has a mono $C_4$ is on my slides:

   `https://www.cs.umd.edu/~gasarch/COURSES/752/S26/slides/C4talk.pdf`

   (It is not due to me.)

   The proof is a lot of case work. It starts with a mono $K_3$. Is there a proof with less casework than starts with two mono $K_3$'s?

## 5 PROJECT FIVE: Cycles

$R(C_4) = 6$ is known. The proof that there is a COL: $\binom{[5]}{2} \rightarrow [2]$ with no mono $C_4$ is the same COL: $\binom{[5]}{2} \rightarrow [2]$ with no mono $K_3$.

Here is the odd thing: That coloring has a mono $C_5$.

Note a different between $K_k$'s and $C_k$'s:

If a graph has a mono $K_k$ then it has a mono $K_{k-1}$.

If a graph has a mono $C_k$ then it does not need to have a mono $C_{k-1}$.

This leads to a new problem:

$R(C_{\le k})$ is the least $n$ such that for all COL: $\binom{[n]}{\rightarrow}[n]$ there exists either a mono $C_k$ or a mono $C_{k+1}$ or $\cdots$ or mono $C_n$.

I think this is a new notion. Study it!

## 6 PROJECT FOUR: Rado's Theorem for Equations that Do Not fit Rado's Theorem

**Rado's Theorem** Let $a_1, \ldots, a_k \in \mathbb{Z}$. Let $E(a_1, \ldots, a_k)$ be the equation
$\sum_{i=1}^{k} a_i x_i = 0$.

Assume that some subset of the $a_i$'s sums to 0. For all $c$, there exists $R = RADO(a_1, \ldots, a_k; c)$ such that, for all COL$[n] \rightarrow [c]$ there is a monochromatic solution to $E(a_1, \ldots, a_k)$. Unless the equation is $x = y$ one can make the numbers in the solution distinct.

Assume that no subset of the $a_i$'s sums to 0.

Let $B = \{|\sum_{i \in I} a_i| \colon I \subseteq [k]\}$.

Let $p$ be the least prime that is larger than every element of $B$. Then there exists COL: $\mathbb{N} \rightarrow [p]$ with no mono solution.

**End of Statement of Rado's Theorem**

1. Consider $x + 2y - 4z = 0$. The set $B$ is $\{1, 2, 3, 4\}$. Hence there is a coloring COL$\mathbb{N} \rightarrow [5]$ with no mono solution.

   What about 2-colorings? 3-colorings? 4-colorings?

   This question can be asked for any linear equation for which $0 \notin B$.

   One can ask both about distinct solutions or allow non-distinct solutions.

2. There is some literature on allowing a constant, for example $x + y - z = 10$. Survey and expand it.

3. There is some literature on non-linear equations. This is very hard. Some empirical studies may work here. See next point.

4. It is known that

   a) For all 2-colorings of [7825] there is an $x, y, z$ the same color with $x^2 + y^2 = z^2$.

   b) There is a 2-coloring of [7824] with no $x, y, z$ the same color with $x^2 + y^2 = z^2$.

   This was done with a massively big SAT solver.

QUESTION: Try to find a coloring of $[n]$ with no $x, y, z$ etc.

You won't get anywhere near 7824, but see how far you can get.

## 7 Using Hard Math

1. Roth's Theorem and some other theorems use Fourier analysis. The intuition is lost on me since I don't know Fourier analysis. A paper where you do an intro to Fourier analysis the normal way (e.g., the heat equation) and then Roth's theorem so one has an idea of how Fourier analysis is being used.

2. Same with Ergodic Theory, ultrafilter proofs, topological proofs.

3. Poly VDW theorem allowing non-zero constant terms.