

The Circle Crossing Game

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Introduction

The circle crossing game is a game that is first set up with a circle with n points on it. (For this problem we have assumed they are equally spaced). There are two players: Player 1 and Player 2. Player 1 and 2 take turns picking two of the points and drawing a line between them. Player 1 begins first. The first player to draw a line that crosses another line loses. There must be at least 3 points on the circle when the game begins. Chords cannot be repeated, shared endpoints are allowed, and a player with no legal move loses.

Research Objective

The objective of this research project was to find two things: (1) For what values of n will player 1 win or lose assuming perfect play. (2) What does perfect play look like for each of two scenarios: player 1 should win with best play or player 2 should win with best play. It is theoretically impossible for there to be a draw in this game.

Findings

Method

In working to solve the problem our team worked to find two things: a rigorous mathematical proof, and a simulation set up to find the instances of n a player loses or wins. While playing around with the problem I stumbled on something promising, it seemed that the problem could be solved using simple mathematical logic. Soon I had a sound proof that seemed to solve the problem outright: Player 1 always wins. One of my teammates was soon able to complete a simulation of 10000 runs containing shapes with 6, 7, 8, and 9 sides. We did not do

further test runs with more sides as we believed our range was sufficient to confirm our proof, and the amount of run time required was not suitable and too slow for our purposes.

Results

(i) Mathematical Proof

Theorem: *Player 1 has a winning strategy in the Circle Crossing Game for all values of $n \geq 3$*

Lemma 1: *The maximum amount of noncrossing lines, chords that can be drawn on points arranged around a circle, contains exactly $2n - 3$ chords when n is 3 or greater*

Proof: First and foremost, a boundary chord, meaning a chord connecting two adjacent points, is unable to cross another chord in all situations. Therefore, the maximum amount of moves must be at least n moves.

Next we will deal with any other possible lines that can be drawn within the boundary. It is clear that when the theoretical maximum of lines drawn are reached every ensuing shape within the boundary chords should be triangles. If any shape were to have more than 3 sides, at least one other line could be drawn inside that region. Therefore, the maximum configuration is based on the triangulation of a convex n -gon.

A convex n -gon contains exactly $n - 2$ triangles and $n - 3$ interior diagonals. This can be proven by using logic. Pick any vertex v in a given polygon with n sides. There are now $n - 1$ vertices available to connect to. However, vertex v is guaranteed to have two adjacent vertices, connecting them would only lead to redrawing the side of a polygon. This leads to the number of possible vertices to connect to $n - 1 - 2$ or $n - 3$ vertices, which follows that only $n - 3$ diagonals

can be drawn. Every noncrossing diagonal drawn splits the original region into two proving that there are $(n - 3) + 1$ or $n - 2$ total possible regions, which we proved earlier are simply triangles.

Thus the maximum number of chords that can be drawn are exactly $2n - 3$ chords.

Lemma 2: *It is not possible to interfere with the maximum number of chords that can be drawn assuming perfect play from both sides. If the maximum number of lines that can be drawn is 'm', every line drawn will only decrease the number of possible lines by 1.*

Proof:

(i) Let us take a polygon with n sides, we will split the polygon using a chord into parts with m sides and $n - m$ sides. This creates two new polygons of $m + 1$ sides and $n - m + 1$ sides.

(ii) As discussed previously, the maximum amount of non intersecting diagonals that can be drawn in a convex polygon is the *# of sides* - 3. This results in the total possible amount of possible diagonals that can be drawn $(n - m + 1) - 3 = n - m - 2$ possible lines for the first region and $(m + 1) - 3 = m - 2$ possible lines.

(iii) Now combining the number of possible diagonals from each region gives $(n - m - 2) + (m - 2) = n - 4$ possible lines, which is one less diagonal to draw.

Therefore any polygon that is split into two polygons by a diagonal will always result in a total of one less possible maximum number of lines that can be drawn. Therefore, there is no way to avoid the maximum configuration of $2n - 3$ lines drawn, assuming perfect play from both sides.

Proof of Theorem: *Player One has a clear winning strategy for every possible value of n : simply draw a line that does not immediately lose the game, or in other words intersect any chord already on the map.*

Proof: By lemma 1, there is always a maximum configuration of $2n - 3$ non-intersecting chords playable for a game with n points. By lemma 2, there is no way to stop the maximum configuration of $2n - 3$ non-intersecting chords being drawn on the board unless either player makes a mistake. $2n - 3$ is always an odd number, therefore player 1 will always get the last move to complete the maximal configuration of lines. Therefore as long as player 1 does not play a line that intersects another chord, the only way for the game to end earlier is if player 2 draws an intersecting line. Thus, player 1's strategy is to simply play logically, safe moves to always guarantee to win the game.

(ii) Computational Verification

Once the proof was completed we decided it was prudent to run many simulation games to see whether the simulations supported the proof. Sidhanth was able to create an application through python that simulated games and recorded whether which player won or lost. We ran 10000 simulated games for each of the values of $n = 6, 7, 8$, and 9. Player One, the player who always moves first, won every single game without fail.

Personal Reflection

This project was enlightening to me in many ways. The initial goal of this project was to build and train an Artificial Intelligence (AI) model to try to derive patterns to solve complex games based on math and logic. However, in first trying to tackle the problem manually as well as running many simulations we were able to quickly derive a solution without the need to train any model at all. To me this was a lesson in the usage of AI in the real world. AI is a powerful tool that is only getting stronger, and neural networks are powerful tools to quickly derive patterns and predictions based on a very complex set of data. However, we would generally not

use a neural network to find the best fitting line on a small dataset of (x,y) points. This project has taught me the importance of looking back and first truly understanding the problem, its context, and its scope before automatically looking to a more powerful tool. It was insightful in light of how this age likes to predict a future where human thinking will no longer be necessary and AI will be able to do everything that humans can and more. But this project has taught me the importance of always thinking for myself, and treating AI as another tool and not an always correct solution.

This project was a great boost for my mathematical knowledge as well. The greatest moment I had in this project was the rush of excitement I felt as I had that epiphany on how I would solve the problem. I remember playing out the proof in my head, bringing out a piece of paper and just formulating the proof, constantly looking for holes. I have always been highly interested in machine learning and AI, and this project brought it all together. Although it was never necessary to train an actual neural network on this problem, just helping to create the simulation and writing the proof helped me connect the worlds of mathematical logic with software engineering. Math is about collaboration. This project was much more efficient and motivating due to having multiple team members who supported the project from different angles, confirming and evaluating each other's work. Dr. Gasarch was an important supporter who taught me a lesson on the general approach of solving a math problem through training an AI. An AI able to learn the game perfectly after tens of thousands of simulations in it of itself is not useful for humans to understand the game. Dr. Gasarch continually pushed our group to derive a humanly understandable strategy from simulation results that are both clear and justifiable.

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