Exact Rado Numbers for Polynomial Difference Equations

Ishan Kumthekar[1] and William Gasarch[2]

[1]Department of Mathematics, University of Florida, Gainesville, FL 32611, USA,
ishankumthekar@ufl.edu
[2]Department of Computer Science, University of Maryland, College Park, MD 20742, USA,
gasarch@umd.edu

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Abstract

We consider exact two-color Rado numbers for nonlinear equations of the form $x - y = aP(z)$, where $P(z)$ is an integer-valued polynomial satisfying $P(0) = 0$ and $P(1) = 1$. Doss, Saracino, and Vestal proved that $\mathcal{R}_2(x - y = z^d) = 2^{d+1} + 1$ for $d \geq 2$. Since the Polynomial Van der Waerden number satisfies $W(z^d, 2) = 2^d + 1$, their formula can be rewritten as

$$\mathcal{R}_2(x - y = P(z)) = P(P(1) + 1) + W(P(z), 2)$$

when $P(z) = z^d$. This suggests a structural connection between Polynomial Van der Waerden and Rado numbers. Motivated by this identity, we ask for which integer-valued polynomials $P(z)$ and positive integers a the analogous formula

$$\mathcal{R}_2(x - y = aP(z)) = aP(a + 1) + W(aP(z), 2)$$

continues to hold. We give general conditions answering this question and show that they extend the earlier result of Doss, Saracino, and Vestal while producing several new infinite families of exact nonlinear Rado numbers. These include exact formulas for equations such as $x - y = c(z^m + z^n)$ and $x - y = a\sum_{r=1}^{z} r^d$, along with families arising from polygonal numbers, simplex numbers, and rising factorials.

1 Introduction

Schur's theorem states that every finite coloring of the positive integers contains monochromatic positive integers x, y, z satisfying

$$x + y = z.$$

The Schur number $S(r)$ is the largest N for which $[N] = \{1, \ldots, N\}$ has an r-coloring with no monochromatic solution. The first values are

$$S(1) = 1, \qquad S(2) = 4, \qquad S(3) = 13.$$

Rado extended Schur's theorem to homogeneous linear equations and systems [7]. The finite form leads to Rado numbers.

Definition 1.1. Let E be an equation and let $c \geq 1$. The *c-color Rado number* of E, denoted by

$$\mathcal{R}_c(E),$$

is the least positive integer N such that every c-coloring of $[N]$ contains a monochromatic solution to E. Variables in the solution need not be distinct.

Example 1.2. Some two-color Rado numbers for linear equations, due to Beutelspacher and Brestovansky [2], are

$$\begin{aligned} \mathcal{R}_2(x_1 + \cdots + x_{m-1} = x_m) &= m^2 - m - 1, \qquad (m \geq 3), \\ \mathcal{R}_2(x_1 + x_2 + x_3 = x_4) &= 11. \end{aligned}$$

Other examples with nonlinear equations are given by Doss, Saracino, and Vestal [3]. After rearranging variables,

$$\begin{aligned} \mathcal{R}_2(x - y = z^d) &= 2^{d+1} + 1, \qquad (d \geq 2), \\ \mathcal{R}_2(ax - y = z^2) &= a - 1, \qquad (a \geq 2). \end{aligned}$$

Definition 1.3. Let $P(z)$ take positive integer values at positive integers. The *c-color Polynomial Van der Waerden number*,

$$W(P(z), c),$$

is the least positive integer N such that every c-coloring of $[N]$ contains integers $u < v$ colored the same and satisfying

$$v - u = P(t)$$

for some positive integer t.

Example 1.4. For $P(z) = z^d$ with $d \geq 2$,

$$W(z^d, 2) = 2^d + 1.$$

To show $W(z^d, 2) > 2^d$, color $[2^d]$ by parity. For $z = 1$, we have $z^d = 1$, but adjacent integers have different colors. For $z \geq 2$, we have $z^d \geq 2^d$ while no two integers in $[2^d]$ are at distance 2^d or more. Hence there is no monochromatic dth-power difference in $[2^d]$.

For the upper bound, consider any 2-coloring of $[2^d + 1]$. If two adjacent integers have the same color, they differ by $1 = 1^d$. Otherwise the coloring alternates, so 1 and $2^d + 1$ have the same color. They differ by 2^d, giving a monochromatic dth-power difference. Therefore, $W(z^d, 2) \leq 2^d + 1$.

For $P(z) = z^d$, we have $P(1) = 1$. Thus, the Doss–Saracino–Vestal formula

$$\mathcal{R}_2(x - y = z^d) = 2^{d+1} + 1 = 2^d + 1 + 2^d$$

can be written as

$$\mathcal{R}_2(x - y = P(z)) = P(P(1) + 1) + W(P(z), 2).$$

This raises the natural question of when such an identity persists beyond the power polynomials $P(z) = z^d$.

2 Polynomial Van der Waerden numbers

Assume from now on that $P(z)$ is integer-valued,

$$P(0) = 0, \qquad P(1) = 1, \tag{1}$$

and

$$1 = P(1) < P(2) < P(3) < \cdots . \tag{2}$$

Let

$$h = \min\{j \geq 2 : P(j) \text{ is even}\}.$$

Thus $P(1), \dots, P(h-1)$ are odd and $P(h)$ is even.

Lemma 2.1. *If $P(z)$ has degree $d \geq 1$, then h is well-defined and*

$$2 \leq h \leq d + 1.$$

Proof. Since $P(1) = 1$ is odd, we have $h \geq 2$. Suppose, for contradiction, that $P(2), \dots, P(d+1)$ are all odd. Together with $P(0) = 0$ and $P(1) = 1$, the parities of $P(0), P(1), \dots, P(d+1)$ are therefore $0, 1, 1, \dots, 1$.

Recall that $\Delta P(n) = P(n+1) - P(n)$. Starting from this parity sequence, the first forward differences modulo 2 are $1, 0, 0, \dots, 0$. Taking differences again gives the same pattern, with one fewer entry, since $0 - 1 \equiv 1 \pmod 2$ and all remaining differences are 0. Repeating this process, after $d + 1$ differences only the 1 remains. Hence

$$\Delta^{d+1} P(0) \equiv 1 \pmod 2.$$

But each forward difference lowers the degree by one, so the $(d+1)$st forward difference of a degree-d polynomial is identically zero. This is a contradiction. Therefore some $P(j)$ with $2 \leq j \leq d+1$ is even, and hence $h \leq d + 1$. ☐

Lemma 2.2. *Under* (1) *and* (2),

$$W(P(z), 2) = P(h) + 1.$$

Proof. Color $[P(h)]$ by parity. If $j < h$, then $P(j)$ is odd, so two integers differing by $P(j)$ have opposite parity and therefore cannot have the same color. If $j \geq h$, then $P(j) \geq P(h)$ and cannot occur inside $[P(h)]$. Hence $W(P(z), 2) > P(h)$.

Conversely, consider any coloring of $[P(h) + 1]$. If two consecutive integers have the same color, their difference is $P(1) = 1$. Otherwise the colors alternate. Since $P(h)$ is even, 1 and $P(h) + 1$ then have the same color and differ by $P(h)$. Thus $W(P(z), 2) \leq P(h) + 1$. ☐

Lemma 2.3. *For every positive integer a,*

$$W(aP(z), 2) = a\big(W(P(z), 2) - 1\big) + 1$$

Consequently,

$$W(aP(z), 2) = aP(h) + 1$$

Gasarch, Kruskal, Kruskal, and Price prove the first identity for $P \in \mathbb{Z}[z]$ [4]. Their proof applies unchanged to integer-valued polynomials. The second identity follows from Lemma 2.2.

3 Main theorem

Theorem 3.1. *Let $P(z)$ be an integer-valued polynomial satisfying* (1) *and* (2), *and let*

$$h = \min\{j \geq 2 : P(j) \text{ is even}\}.$$

Let $a \geq h$, and assume

$$P(2a+1) \geq P(a+1) + P(h). \tag{3}$$

Then

$$\boxed{\mathcal{R}_2(x - y = aP(z)) = aP(a+1) + W(aP(z), 2)} \tag{4}$$

if and only if there is no j with $1 \leq j \leq 2a$ such that

$$0 < |P(j) - P(a+1)| < P(h) - 1 \quad \text{and} \quad |P(j) - P(a+1)| \text{ is odd.} \tag{5}$$

We first prove the upper bound.

Lemma 3.2. *Under the assumptions of Theorem 3.1,*

$$\mathcal{R}_2(x - y = aP(z)) \leq aP(a+1) + aP(h) + 1.$$

Proof. The following upper bound argument generalizes the construction of Doss, Saracino, and Vestal [3]. Their argument uses the distances 1 and 2^d, while here we replace these with general distances $aP(1) = a$ and $aP(h)$. Set $N = aP(a+1) + aP(h) + 1$ and suppose that $[N]$ has no monochromatic solution. We first show that 1 and h have the same color. After interchanging colors if necessary, suppose that 1 is red and h is blue, and put $H = P(h)$. Consider

$$u_i = 1 + ia, \qquad 0 \leq i \leq 2H.$$

These all lie in $[N]$, since $H < P(a+1)$. No two consecutive u_i can both be red, since their difference is $a = aP(1)$. Also, if u_i is blue for $0 \leq i \leq H$, then u_{i+H} must be red, since their difference is $aH = aP(h)$.

Now u_0 is red, so u_1 is blue, hence u_{H+1} is red. Thus u_H is blue, and hence u_{2H} is red. Moreover, for each $1 \leq i \leq H - 1$, at least one of u_i and u_{i+H} is red. Therefore there are at least

$$(H - 1) + 2 = H + 1$$

red vertices among $u_0, \ldots, u_{2H}$. Since no two consecutive vertices are red, this is the maximum possible number, forcing the red vertices to be

$$u_0, u_2, u_4, \ldots, u_{2H}.$$

But $H = P(h)$ is even, so u_H is red, contradicting that u_H is blue. Hence 1 and h have the same color. Without loss of generality assume they are red.

Suppose now that some $z \leq a + 1$ is blue. Consider the two rows

column	0	1	2	$\cdots$	$P(h) - 1$	$P(h)$
top	1	$1 + a$	$1 + 2a$	$\cdots$	$1 + (P(h) - 1)a$	$1 + aP(h)$
bottom	$1 + aP(z)$	$1 + aP(z) + a$	$1 + aP(z) + 2a$	$\cdots$	$1 + aP(z) + (P(h) - 1)a$	$1 + aP(z) + aP(h)$

Each row is a copy of the odd cycle whose consecutive edges have length $aP(1) = a$ and whose closing edge has length $aP(h)$. The two entries in each column differ by $aP(z)$, so they cannot both be blue. Hence each column contains a red entry.

There are now two possibilities.

1. Two adjacent columns contain red entries in the same row. Then those two red entries differ by $a = aP(1)$, so together with the red root 1 they give a red solution.
2. Otherwise each column has exactly one red entry, and these red entries alternate between the two rows. Since $P(h)$ is even, columns 0 and $P(h)$ contain red entries in the same row. These two entries differ by $aP(h)$, so together with the red root h they give a red solution.

Both cases are impossible. Therefore every $z \le a+1$ is red. In particular, 1 and $a+1$ are red, but

$$(a+1) - 1 = a = aP(1),$$

so $(a+1, 1, 1)$ is a red solution, a contradiction. □

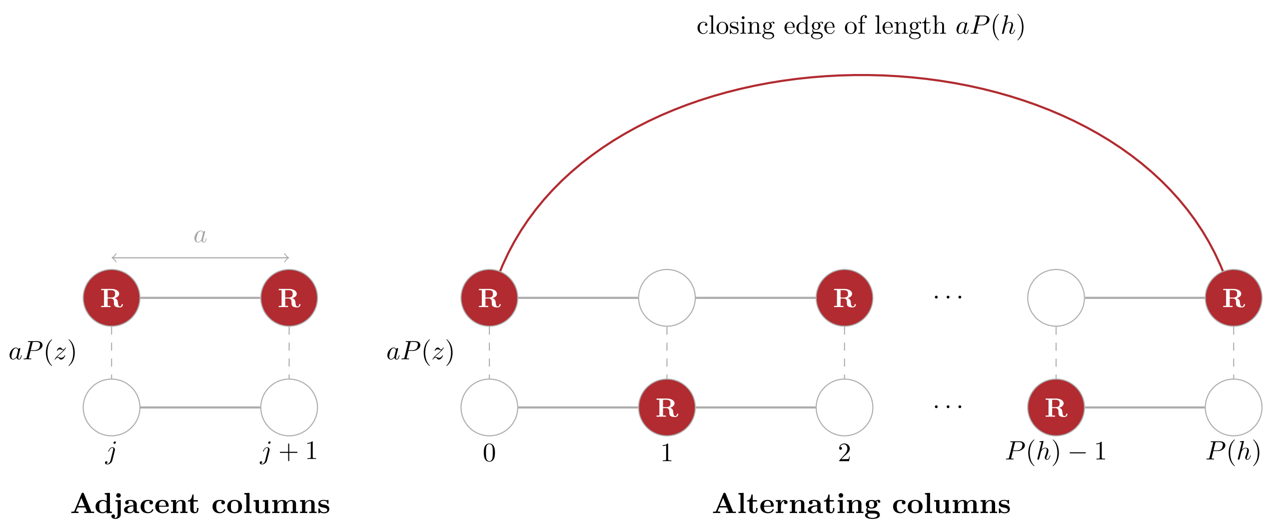

Figure 1: The two possible arrangements of the selected red entries.

Lemma 3.3. *Under the assumptions of Theorem 3.1, suppose*

$$[a(P(a+1) + P(h))]$$

has no monochromatic solution. After interchanging colors if necessary,

$$1, \dots, a \text{ are red}, \qquad a+1, \dots, 2a \text{ are blue}.$$

Moreover, for $0 \le i \le P(h) - 1$,

$$1 + ia \text{ is red} \iff i \text{ is even},$$

$$1 + aP(a+1) + ia \text{ is red} \iff i \text{ is odd},$$

and $1 + aP(h)$ *is blue.*

Proof. Using the proof of Lemma 3.2, the roots 1 and h have the same color since $1 + 2aP(h) \le a(P(a+1) + P(h))$. Assume they are red. If some $z \le a$ were blue, the argument from Lemma 3.2 would give a red solution. The construction still fits inside the present interval because $1 + aP(h) + aP(z) \le 1 + aP(h) + aP(a) \le a(P(a+1) + P(h))$. Therefore $1, \dots, a$ are red.

Now let $a+1 \le z \le 2a$. Then $z-a \in [1,a]$, so $z-a$ and 1 are red. If z were also red, then $z-(z-a)=a=aP(1)$, and $(z, z-a, 1)$ would be a red solution. Therefore $a+1, \dots, 2a$ are blue.

Consider the two rows

column	0	1	$\cdots$	$P(h)-1$	$P(h)$
upper row	$1+aP(a+1)$	$1+aP(a+1)+a$	$\cdots$	$1+aP(a+1)+a(P(h)-1)$	
lower row	1	$1+a$	$\cdots$	$1+a(P(h)-1)$	$1+aP(h)$

The lower row has $P(h)+1$ vertices. Consecutive vertices differ by $a=aP(1)$, while its first and last vertices differ by $aP(h)$. These edges form an odd cycle. Since the roots 1 and h are red, no edge of this cycle can have two red endpoints. Blue-blue endpoints are allowed, since the roots 1 and h corresponding to these edge lengths are red. Since $P(h)$ is even, at most $\lfloor \frac{P(h)+1}{2} \rfloor = \frac{P(h)}{2}$ vertices in the lower row are red. The upper row consists of $P(h)$ vertices whose consecutive vertices also differ by $a=aP(1)$. Since root 1 is red, no two consecutive vertices can both be red. Thus at most every other vertex can be red, so, since $P(h)$ is even, the upper row contains at most $\frac{P(h)}{2}$ red vertices.

Now look at the vertical columns. For every $0 \le i \le P(h)-1$, the two entries in column i differ by $aP(a+1)$. Since the root $a+1$ is blue, the two entries in a column cannot both be blue. Thus each of the $P(h)$ columns contains at least one red vertex.

The two rows contain at most $P(h)$ red vertices altogether, while the $P(h)$ columns each require at least one red vertex. Therefore every column contains exactly one red vertex, the two entries in each column have opposite colors, and the unpaired final vertex $1+aP(h)$ is blue.

For $0 \le i \le P(h)-2$, two consecutive vertices in the lower row cannot both be red, since they differ by $a=aP(1)$ and the root 1 is red. They also cannot both be blue, since then the two vertices directly above them would both be red and would differ by $a=aP(1)$. Hence the first $P(h)$ vertices of the lower row alternate in color.

Its first vertex is 1, which is red. Therefore

$$1+ia \text{ is red} \iff i \text{ is even} \qquad (0 \le i \le P(h)-1).$$

The unpaired blue vertex is not part of this alternation, so the final horizontal edge may have two blue endpoints. The upper row has the opposite colors, proving the lemma. □

Proof of Theorem 3.1. Put

$$N = a\big(P(a+1)+P(h)\big)+1.$$

By Lemmas 2.2 and 2.3, $W(aP(z),2)=aP(h)+1$. Hence the right side of (4) is exactly N, and Lemma 3.2 gives

$$\mathcal{R}_2(x-y=aP(z)) \le N.$$

Thus the question is whether this upper bound is sharp. By the definition of the Rado number,

$$\mathcal{R}_2(x-y=aP(z)) = N$$

if and only if $[N-1]$ has a coloring with no monochromatic solution. We prove that such a coloring exists if and only if no j satisfies (5).

First suppose that $[N-1]$ has a coloring with no monochromatic solution. By Lemma 3.3, after interchanging colors if necessary, $1,\dots,a$ are red and $a+1,\dots,2a$ are blue. Moreover, for $0 \le i \le P(h)-1$,

$$1+ia \text{ is red} \iff i \text{ is even}, \qquad 1+aP(a+1)+ia \text{ is red} \iff i \text{ is odd}.$$

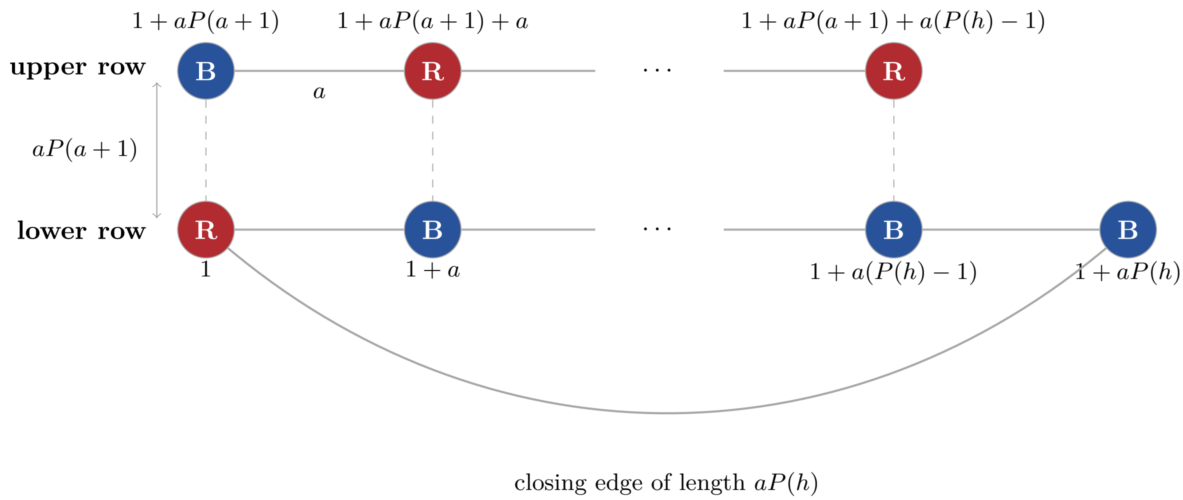

Figure 2: The lower row becomes an odd cycle by adding the edge of length $aP(h)$.

We show that no j satisfies (5). Suppose instead that some $1 \le j \le 2a$ does. Then

$$r = \lvert P(j) - P(a+1) \rvert$$

is positive and odd, with $r < P(h) - 1$. Since $P(h)$ is even, this means $r \le P(h) - 3$.

1. Suppose $1 \le j \le a$. Since P is increasing, $r = P(a+1) - P(j)$. The number $r+1$ is even, so $1 + a(r+1)$ is red. Also $1 + aP(a+1) + a$ is red, since it corresponds to the odd index 1 in the upper row of Lemma 3.3. Their difference is

$$(1 + aP(a+1) + a) - (1 + a(r+1)) = a(P(a+1) - r) = aP(j).$$

   The root j is also red because $j \le a$. Hence these three integers give a red solution, a contradiction.

2. Suppose $a+1 \le j \le 2a$. Since $r > 0$, we have $j > a+1$, and therefore $r = P(j) - P(a+1)$. The number $1+a$ is blue. Since $r+1$ is even, $1 + aP(a+1) + a(r+1)$ is also blue. Their difference is

$$(1 + aP(a+1) + a(r+1)) - (1+a) = a(P(a+1) + r) = aP(j).$$

   The root j is blue because $a+1 \le j \le 2a$. Hence these three integers give a blue solution, again a contradiction.

Therefore the absence of any j satisfying (5) is necessary.

We now prove the converse. Assume that no j satisfies (5). We construct a coloring of $[N-1]$ with no monochromatic solution. This construction extends the alternating end coloring used in [3].

Divide $[N-1]$ into consecutive blocks of length a:

$$B_i = \{(i-1)a + 1, \dots, ia\}, \qquad 1 \le i \le P(a+1) + P(h).$$

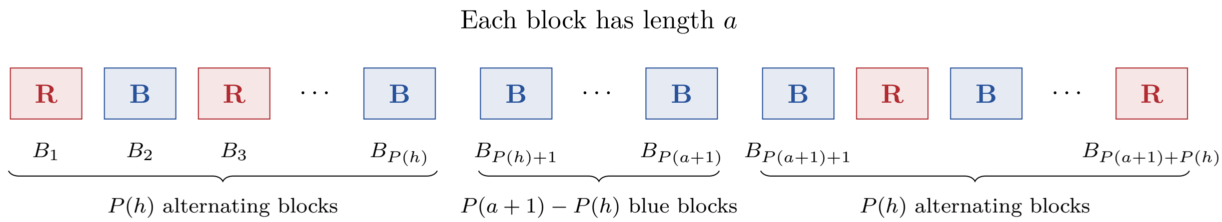

Figure 3: A coloring of $[N-1]$ with no monochromatic solution

Color the blocks as follows. For $1 \le i \le P(h)$, color B_i red when i is odd and blue when i is even. Color $B_{P(h)+1}, \ldots, B_{P(a+1)}$ blue. Finally, for $1 \le i \le P(h)$, color $B_{P(a+1)+i}$ blue when i is odd and red when i is even. This coloring is shown in Figure 3.

We first reduce the problem to the block indices. Suppose $x \in B_i$ and $y \in B_k$. Write $x = (i-1)a + s$ and $y = (k-1)a + t$, where $1 \le s, t \le a$. If $x - y = aP(z)$, then

$$a(i-k) + (s-t) = aP(z).$$

Thus $s - t$ is divisible by a. Since $|s-t| < a$, we must have $s = t$, and consequently $i - k = P(z)$. Therefore a difference $aP(z)$ occurs exactly between equal positions in two blocks whose indices differ by $P(z)$. To rule out a monochromatic solution with root z, it is enough to show that no two blocks having the color of z have index difference $P(z)$.

We consider three possible ranges for z.

1. Let $1 \le z \le a$. Then $z \in B_1$, so the root z is red. We must show that no two red blocks have index difference $P(z)$.

   First consider two red blocks lying in the same alternating end. Their indices have the same parity, so their separation is even and strictly less than $P(h)$. This cannot equal $P(z)$. If $z < h$, then $P(z)$ is odd by the definition of h. If $z \ge h$, then $P(z) \ge P(h)$ because P is increasing.

   It remains to consider a red block from the first alternating end and a red block from the final alternating end. A red block in the first end has index i, where i is odd, while a red block in the final end has index $P(a+1) + k$, where k is even.

   Since $z \le a$, we have $P(z) < P(a+1)$. Thus if these two blocks were separated by $P(z)$, then

   $$P(a+1) + k - i = P(z),$$

   so

   $$P(a+1) - P(z) = i - k.$$

   The number $i - k$ is positive and odd. Moreover, since $i \le P(h) - 1$ and $k \ge 2$,

   $$1 \le i - k \le P(h) - 3.$$

   This is precisely one of the values excluded by (5). Therefore no two red blocks are separated by $P(z)$, and no red solution can occur.

2. Let $a+1 \le z \le 2a$. Then $z \in B_2$, so the root z is blue. Since P is increasing, write

$$P(z) = P(a+1) + r, \qquad r \ge 0.$$

If $r \ge P(h)$, then $P(z) \ge P(a+1) + P(h)$, which is at least the total number of blocks. Such a block separation cannot occur.

Suppose instead that $0 \le r < P(h)$, and suppose two blue blocks are separated by $P(z) = P(a+1) + r$. If the first block is B_i, then the second must be

$$B_{P(a+1)+r+i}.$$

Since there are only $P(a+1) + P(h)$ blocks altogether, we must have $i \le P(h) - r$. In particular, the first block lies in the initial alternating end, and the second lies in the final alternating end.

In the initial end, B_i is blue exactly when i is even. In the final end, $B_{P(a+1)+k}$ is blue exactly when k is odd. Here $k = r + i$. Thus, if both blocks are blue, then i is even and $r + i$ is odd. Hence r must be odd.

Since i is a positive even integer, $i \ge 2$. Together with $i \le P(h) - r$, this gives $r \le P(h) - 2$. As r is odd and $P(h)$ is even, in fact

$$1 \le r \le P(h) - 3.$$

But then $|P(z) - P(a+1)| = r$ is a positive odd integer smaller than $P(h) - 1$, contrary to (5). Therefore no blue solution can occur.

3. Let $z \ge 2a + 1$. By (3),

$$P(z) \ge P(2a+1) \ge P(a+1) + P(h).$$

Hence

$$aP(z) \ge a\big(P(a+1) + P(h)\big) = N - 1.$$

But the difference between two integers in $[N-1]$ is at most $N-2$. Thus no pair in $[N-1]$ can have difference $aP(z)$.

The coloring of Figure 3 therefore contains no monochromatic solution. Hence $\mathcal{R}_2(x - y = aP(z)) > N - 1$, and so $\mathcal{R}_2(x - y = aP(z)) \ge N$. Together with the upper bound $\mathcal{R}_2(x - y = aP(z)) \le N$, we obtain

$$\mathcal{R}_2(x - y = aP(z)) = N = aP(a+1) + aP(h) + 1 = aP(a+1) + W(aP(z), 2).$$

This proves Theorem 3.1. $\square$

4 Simplifying the hypotheses

The exact criterion in Theorem 3.1 can be tedious to verify separately for each polynomial family. Beyond $a \ge h$, the two conditions to check are

1. The range condition
$$P(2a+1) \geq P(a+1) + P(h)$$
2. The absence of a $1 \leq j \leq 2a$ for which
$$0 < |P(j) - P(a+1)| < P(h) - 1$$
and $|P(j) - P(a+1)|$ is odd.

The next two corollaries replace these conditions by simpler hypotheses on the consecutive gaps of $P(z)$.

Corollary 4.1. *Let $P(z)$ be an integer-valued polynomial satisfying $P(0) = 0$, $P(1) = 1$, and*
$$1 = P(1) < P(2) < P(3) < \cdots .$$
Suppose that its consecutive gaps strictly increase:
$$P(n+2) - P(n+1) > P(n+1) - P(n) \qquad (n \geq 1).$$
Then for every $a \geq h$,
$$P(2a+1) \geq P(a+1) + P(h).$$
Thus the range condition in Theorem 3.1 is satisfied.

Proof. Write $\Delta_n = P(n+1) - P(n)$. Since the Δ_n are strictly increasing integers, $\Delta_{n+1} \geq \Delta_n + 1$. Applying this a times gives
$$\Delta_{a+j} \geq \Delta_j + a.$$
Since $a \geq h$,
$$\begin{aligned} P(2a+1) - P(a+1) &\geq P(a+h) - P(a+1) \\ &= \sum_{j=1}^{h-1} \Delta_{a+j} \\ &\geq \sum_{j=1}^{h-1} (\Delta_j + a) \\ &= P(h) - 1 + a(h-1) \\ &\geq P(h). \end{aligned}$$
Hence $P(2a+1) \geq P(a+1) + P(h)$. $\square$

After Corollary 4.1, only the forbidden odd differences in (5) remain to be checked. The next corollary replaces that condition by a single inequality.

Corollary 4.2. *Let $P(z)$ be an integer-valued polynomial satisfying $P(0) = 0$, $P(1) = 1$, and*
$$1 = P(1) < P(2) < P(3) < \cdots ,$$
and suppose that its consecutive gaps strictly increase. If $a \geq h$ and
$$P(a+1) - P(a) \geq P(h) - 2,$$
then
$$\mathcal{R}_2(x - y = aP(z)) = aP(a+1) + W(aP(z), 2).$$

Proof. By Corollary 4.1, the range condition already holds. It remains to verify that no j satisfies (5).

If $j \le a$, then
$$P(a+1) - P(j) \ge P(a+1) - P(a) \ge P(h) - 2.$$
Since $P(h)$ is even, every positive odd integer smaller than $P(h)-1$ is at most $P(h)-3$. Thus no $j \le a$ gives a forbidden difference.

If $j \ge a+2$, then the strictly increasing gaps give
$$P(a+2) - P(a+1) > P(a+1) - P(a) \ge P(h) - 2.$$
The gaps are integers, so $P(a+2) - P(a+1) \ge P(h) - 1$, and every later value $P(j)$ is still farther from $P(a+1)$. Thus no $j \ge a+2$ gives a forbidden difference. Finally, $j = a+1$ gives difference zero.

Hence no j satisfies (5), and the result follows from Theorem 3.1. $\square$

Thus, in the applications below, it is enough to check that $P(z)$ is integer-valued, has $P(0) = 0$ and $P(1) = 1$, and is increasing. We then determine h, check that the consecutive gaps strictly increase, and verify only
$$P(a+1) - P(a) \ge P(h) - 2 \qquad a \ge h.$$
Corollary 4.2 then gives the exact Rado number. These conditions are sufficient rather than necessary. If the final gap inequality fails, Theorem 3.1 may still apply directly, as we will see for the simplex numbers below.

5 Applications

We now apply Corollary 4.2 to get several infinite families of exact Rado numbers.

5.1 Powers

Let $P(z) = z^d$, where $d \ge 2$. We have $h = 2$ and $P(h) = 2^d$. The consecutive gaps
$$(n+1)^d - n^d$$
strictly increase. For every $a \ge 2$, the gap at a is larger than the first gap $2^d - 1$, and therefore certainly at least $P(h) - 2$. Corollary 4.2 gives
$$\boxed{\mathcal{R}_2(x - y = az^d) = a(a+1)^d + a2^d + 1 \qquad (a \ge 2).}$$
Notice that the case $a = 1$ is the theorem of Doss, Saracino, and Vestal, so the equation above holds for $a \ge 1$.

5.2 Sums of powers

Let p be a prime and let

$$d_1, \dots, d_p \geq 2, \qquad d_1 \equiv d_2 \equiv \cdots \equiv d_p \pmod{p-1}.$$

Consider

$$P(z) = \frac{1}{p} \sum_{i=1}^{p} z^{d_i}.$$

This polynomial is integer-valued. Indeed, if $p \mid z$, every term in the numerator is divisible by p. If $p \nmid z$, Fermat's theorem and the congruence of the exponents imply that all z^{d_i} are congruent modulo p, so their sum is again divisible by p.

We have $P(0) = 0$ and $P(1) = 1$. Also $P(2)$ is even. If $p = 2$, every term in the numerator is divisible by 4, while if p is odd, the numerator is even and division by the odd integer p preserves evenness. Thus $h = 2$. Each monomial z^{d_i} has strictly increasing consecutive gaps, and hence so does $P(z)$.

Apply Corollary 4.2 with $a = pc$, where $c \geq 1$. Since $pc \geq 2 = h$ and the gaps strictly increase, the gap condition holds. Therefore

$$\boxed{\mathcal{R}_2 \left(x - y = c \sum_{i=1}^{p} z^{d_i} \right) = c \sum_{i=1}^{p} \left((pc+1)^{d_i} + 2^{d_i} \right) + 1.}$$

For $p = 2$, the congruence condition on the exponents is vacuous. Thus, for $m, n \geq 2$,

$$\mathcal{R}_2(x - y = c(z^m + z^n)) = c(2c+1)^m + c(2c+1)^n + c2^m + c2^n + 1.$$

For $p = 3$, the exponents need only have the same parity. Hence, whenever $r, s, t \geq 2$ are all even or all odd,

$$\begin{aligned} \mathcal{R}_2(x - y = c(z^r + z^s + z^t)) = c(3c+1)^r + c(3c+1)^s + c(3c+1)^t \\ + c2^r + c2^s + c2^t + 1. \end{aligned}$$

Thus the same construction gives exact Rado numbers for sums of a prime number of powers satisfying the congruence condition above.

5.3 Sums of consecutive powers

Let

$$P(z) = \sum_{r=1}^{z} r^d, \qquad d \geq 1.$$

By Faulhaber's formula, this is an integer-valued polynomial in z. We have $P(0) = 0$ and $P(1) = 1$, while $P(2) = 1 + 2^d$ is odd and $P(3) = 1 + 2^d + 3^d$ is even. Thus $h = 3$ and $P(h) = 1 + 2^d + 3^d$.

The consecutive gaps are

$$P(n+1) - P(n) = (n+1)^d,$$

and strictly increase. For $a \geq 3$,

$$P(a+1) - P(a) = (a+1)^d \geq 4^d \geq 2^d + 3^d - 1 = P(h) - 2.$$

Hence Corollary 4.2 gives

$$\boxed{\mathcal{R}_2 \left(x - y = a \sum_{r=1}^{z} r^d \right) = a \sum_{r=1}^{a+1} r^d + a(1 + 2^d + 3^d) + 1 \qquad (a \geq 3).}$$

5.4 Polygonal numbers

The theorem also gives a single family containing the triangular, square, pentagonal, hexagonal, and higher polygonal numbers. For $s \geq 3$, let

$$P(z) = \frac{(s-2)z^2 - (s-4)z}{2}.$$

The numerator is always even, so $P(z)$ is integer-valued. Also $P(0) = 0$ and $P(1) = 1$. Its consecutive gaps are

$$P(n+1) - P(n) = (s-2)n + 1,$$

and hence strictly increase. Now $P(2) = s$. If s is even, then $h = 2$ and $P(h) = s$. If s is odd, then $P(2)$ is odd while $P(3) = 3(s-1)$ is even, so $h = 3$ and $P(h) = 3(s-1)$.

If s is even and $a \geq 2$, then

$$P(a+1) - P(a) = (s-2)a + 1 \geq s - 2 = P(h) - 2.$$

If s is odd and $a \geq 3$, then

$$P(a+1) - P(a) \geq 3(s-2) + 1 = 3(s-1) - 2 = P(h) - 2.$$

Thus Corollary 4.2 gives

$$\boxed{\begin{aligned} \mathcal{R}_2\left(x - y = a\frac{(s-2)z^2 - (s-4)z}{2}\right) &= \frac{a}{2}\Big((s-2)(a+1)^2 - (s-4)(a+1)\Big) \\ &\quad + \begin{cases} as + 1, & s \text{ even}, \\ 3a(s-1) + 1, & s \text{ odd}. \end{cases} \end{aligned}}$$

This holds for $a \geq 2$ when s is even and $a \geq 3$ when s is odd. This one formula contains exact Rado numbers associated with every classical polygonal number sequence.

5.5 Simplex numbers

For $k \geq 2$, let

$$P(z) = \binom{z+k-1}{k}.$$

Put $q = 2^{\nu_2(k)}$, where $\nu_2(k)$ is the largest nonnegative integer r such that 2^r divides k. By Kummer's theorem in the case $p = 2$, $\binom{k+t}{k}$ is odd exactly when adding k and t in binary creates no carry [5]. Since $q = 2^{\nu_2(k)}$ corresponds to the least nonzero binary digit of k, no carry occurs for $1 \leq t < q$, while adding $t = q$ produces a carry. Hence

$$h = 1 + q, \qquad P(h) = \binom{k+q}{k}.$$

The consecutive gaps are

$$P(n+1) - P(n) = \binom{n+k-1}{k-1},$$

and strictly increase. Therefore Corollary 4.2 applies whenever

$$a \geq 1+q \qquad \text{and} \qquad \binom{a+k-1}{k-1} \geq \binom{k+q}{k} - 2.$$

Under these conditions,

$$\boxed{\mathcal{R}_2\left(x-y = a\binom{z+k-1}{k}\right) = a\binom{a+k}{k} + a\binom{k+q}{k} + 1.}$$

The following example shows that the gap condition in Corollary 4.2 is not necessary. Take $k=4$ and $a=5$. Then

$$h = 5, \qquad P(5) = 70, \qquad P(6) = 126, \qquad P(7) = 210,$$

so

$$P(6) - P(5) = 56 < 68 = P(h) - 2, \qquad P(7) - P(6) = 84.$$

Corollary 4.2 does not apply. However, the consecutive gaps are strictly increasing, so Corollary 4.1 gives the range condition. Moreover, the nearest values to $P(6) = 126$ are at distances 56 and 84, so no $P(j)$ differs from 126 by an odd integer in $\{1, 3, \ldots, 67\}$. Hence the exact condition in Theorem 3.1 holds, and

$$\mathcal{R}_2\left(x-y = 5\binom{z+3}{4}\right) = 981.$$

5.6 Rising factorials

The identity

$$k!\binom{z+k-1}{k} = z(z+1)\cdots(z+k-1)$$

allows the simplex number result to be written as an exact formula for rising factorials. Apply the simplex number formula with $a = ck!$.

For $k \geq 3$ and $c \geq 1$, put $a = ck!$. Since $q = 2^{\nu_2(k)} \leq k$, we have $a \geq 1 + q = h$. For $k = 3$, we have $q = 1$, and the gap condition is immediate since $\binom{8}{2} = 28 > \binom{4}{3} - 2 = 2$. For $k \geq 4$,

$$\binom{ck!+k-1}{k-1} \geq \binom{k!+k-1}{k-1} > \left(\frac{k!}{k-1}\right)^{k-1} \geq 8^{k-1} > 4^k > \binom{2k}{k} \geq \binom{k+q}{k}.$$

Thus the gap condition holds. When $k = 2$ and $c \geq 2$, we have $a = 2c \geq 4 > h = 3$ and $P(a+1) - P(a) = a + 1 \geq 5 > P(h) - 2 = 4$. Hence

$$\boxed{\begin{aligned}&\mathcal{R}_2\left(x-y = c\, z(z+1)\cdots(z+k-1)\right)\\ &\qquad = c\prod_{j=1}^{k}(ck!+j) + c\prod_{j=1}^{k}(q+j) + 1, \qquad q = 2^{\nu_2(k)}.\end{aligned}}$$

This holds for $k \geq 3$, $c \geq 1$, and also for $k = 2$, $c \geq 2$. The case $k = 2, c = 1$ lies outside the hypotheses used here.

6 Relating Polynomial Van der Waerden and Rado numbers

This paper is motivated by the question of when Polynomial Van der Waerden numbers and nonlinear Rado numbers, two seemingly different Ramsey quantities, are linked by an exact formula. The connection with Polynomial Van der Waerden numbers is easiest to see by comparing the two definitions. The number $W(aP(z), 2)$ forces two equally colored integers $x > y$ such that

$$x - y = aP(t)$$

for some positive integer t. For (x, y, t) to be a monochromatic solution of

$$x - y = aP(z),$$

we only need one additional fact: t must have the same color as x and y.

For the polynomials considered here,

$$W(aP(z), 2) = aP(h) + 1.$$

The proof of Theorem 3.1 shows how this extra color condition on t is forced. In the upper bound, the same distances $aP(1) = a$ and $aP(h)$ that determine $W(aP(z), 2)$ are used to force the colors of the possible roots. In the lower bound, the alternating coloring that avoids these polynomial differences is used at the two ends of the constructed coloring.

The remaining term $aP(a + 1)$ records how far apart these two ends are placed. The condition in Theorem 3.1 is exactly what prevents this placement from creating an additional monochromatic polynomial difference. This gives

$$\mathcal{R}_2(x - y = aP(z)) = aP(a + 1) + W(aP(z), 2).$$

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