Exact Values and Bounds for four color Quadratic Polynomial van der Waerden Numbers

Ishan Kumthekar[1], William Gasarch[2], William Regli[2], and Ian Whitehouse[2]

[1]Department of Mathematics, University of Florida, Gainesville, FL 32611, USA, ishankumthekar@ufl.edu
[2]Department of Computer Science, University of Maryland, College Park, MD 20742, USA, gasarch@umd.edu, regli@umd.edu, ianjw@umd.edu

September 5, 2026

Abstract

We determine the exact four color polynomial van der Waerden numbers for

$$x^2+x, \quad 2x^2+x, \quad 3x^2+x, \quad x^2+7x, \quad 2x^2+7x, \quad 3x^2+7x, \quad x^2+9x, \quad 2x^2+9x, \quad 3x^2+11x.$$

The exact values follow from explicit colorings for the lower bounds and combinatorial proofs for the upper bounds. These proofs use restrictions on gaps between equally colored integers and small graphs that force colors to agree. Simulated annealing is used to find several of the lower colorings, each verified directly. We also establish general lower bounds in terms of the coefficients and forbidden distances. For all integers a, b, not both zero, we prove

$$W(ax^2+bx,4) \leq 64(|a|+|b|)+1.$$

This proof uses a fixed finite graph and exhaustive computations. We include the computational methods and compare the general bounds with the exact values.

1 Introduction

Let $P(x) \in \mathbb{Z}[x]$ be nonzero and satisfy $P(0) = 0$. We use the convention of Gasarch, Kruskal, Kruskal, and Price [3]. For $d \in \mathbb{Z}$, the nonzero values $P(d)$ give the forbidden differences.

Definition 1.1. For $k \geq 2$, the Polynomial Van der Waerden number $W(P(x), k)$ is the least positive integer N such that every k coloring of $[N] = \{1, 2, \ldots, N\}$ contains distinct equally colored integers u, v satisfying

$$v - u = P(d)$$

for some $d \in \mathbb{Z}$.

Because u and v may be interchanged, it is convenient to write

$$\mathcal{D}(P) = \{|P(d)| : d \in \mathbb{Z},\ P(d) \neq 0\}.$$

Thus a coloring is valid exactly when no two equally colored integers are separated by a distance in the forbidden distance set $\mathcal{D}(P)$.

For example,

$$\mathcal{D}\Big(x^2+x\Big) = \{2,6,12,20,30,42,56,72,90,\ldots\}.$$

Therefore $W(x^2+x,4)$ is the first N such that every four coloring of $[N]$ has an equally colored pair at one of these distances.

The general Polynomial Van der Waerden theorem guarantees that these numbers are finite [2], but its quantitative bounds are extremely large. Gasarch et al. developed finite constructions for the four color quadratic case. Their bounds include [3]

$$W\Big(x^2,4\Big) \le 84{,}149{,}474{,}894{,}213{,}522, \qquad W\Big(x^2+x,4\Big) \le 12{,}828{,}393{,}907,$$
$$W\Big(x^2+2x,4\Big) \le 66{,}287{,}711{,}296, \qquad W\Big(x^2+3x,4\Big) \le 10{,}952{,}789.$$

The first of these has since been reduced to the exact value $W(x^2,4) = 58$ [4]. Our goal is similar in spirit. We compute several exact values in Section 2 for the four color case.

Sections 3 and 4 give bounds for arbitrary quadratics. Set $A = |a|$, $B = |b|$, and let $s_1 < s_2$ be the two smallest positive forbidden distances of ax^2+bx. We prove

$$\boxed{\max\{4s_1, 2s_2\} + 1 \le W\Big(ax^2+bx,4\Big) \le 64(A+B)+1.}$$

Further lower bounds use the coefficients and repeated blocks of colors. The block construction attains the exact lower bounds for $2x^2+7x$, $3x^2+7x$, and $3x^2+11x$.

We study simulated annealing as a method for discrete optimization, motivated by alternative computational substrates and the Ising and Potts models. Here the energy counts monochromatic forbidden pairs. The search produces finite colorings. Each displayed coloring is then checked directly. The general lower bounds are proved by explicit constructions, and the upper bound uses an exhaustive finite search. Appendix A gives the algorithms and compares the bounds with the exact values.

We will also use the following identity. It is Lemma 4 of Gasarch et al. [3].

Proposition 1.2. *For every positive integer h,*

$$W(hP(x),k) = h(W(P(x),k)-1)+1.$$

2 Exact Values

The same elementary fact is used in the first three upper bounds.

Lemma 2.1. *Every four coloring of*

$$\{0,1,\ldots,16\}$$

contains two equally colored integers whose difference belongs to

$$F = \{1,2,5,7,12,15\}.$$

Proof. Suppose, for contradiction, that there is a four coloring of

$$\{0,1,\ldots,16\}$$

with no two equally colored integers at a difference in F.

There are 17 integers and only four colors. If every color were used at most four times, then at most $4 \cdot 4 = 16$ integers would be colored. Hence some color is used at least five times. Choose five integers of that color and write them in increasing order as

$$a_0 < a_1 < a_2 < a_3 < a_4.$$

For $1 \le i \le 4$, let $g_i = a_i - a_{i-1}$.

By our assumption, no two equally colored integers may differ by an element of F. Since a_{i-1} and a_i have the same color, we must have

$$g_i \notin \{1, 2, 5, 7\}.$$

In particular, $g_i \ge 3$.

Also,

$$g_1 + g_2 + g_3 + g_4 = a_4 - a_0 \le 16.$$

If some $g_i \ge 8$, then the other three gaps are each at least 3, so

$$g_1 + g_2 + g_3 + g_4 \ge 8 + 3 + 3 + 3 = 17,$$

a contradiction. Therefore every $g_i \le 7$.

Hence each gap is an integer between 3 and 7, but it cannot equal 5 or 7. Therefore

$$g_i \in \{3, 4, 6\} \qquad (1 \le i \le 4).$$

We now consider how many of the four gaps are equal to 6. Suppose first that no gap is 6. Then every gap is either 3 or 4. A 3 and a 4 cannot occur next to one another, since two chosen positions would then differ by $3 + 4 = 7$, a forbidden distance. Hence all four gaps are equal. If they are all 3, then $a_4 - a_0 = 3 + 3 + 3 + 3 = 12$, while if they are all 4, then $a_3 - a_0 = 4 + 4 + 4 = 12$. Both cases give two equally colored positions at the forbidden difference 12.

Next, there cannot be two gaps equal to 6, since the remaining two gaps are at least 3, giving $g_1 + g_2 + g_3 + g_4 \ge 6 + 6 + 3 + 3 = 18 > 16$.

It remains to consider exactly one gap equal to 6. If the other three gaps are all 3, then $a_4 - a_0 = 6 + 3 + 3 + 3 = 15$, a forbidden distance. Otherwise one of the remaining gaps is 4. There cannot be two gaps equal to 4, since then $6 + 4 + 4 + 3 = 17 > 16$. Thus the four gaps are $6, 4, 3, 3$ in some order. Since a gap 4 cannot be adjacent to a gap 3, the 4 must be next to the 6 and at one end of the sequence. Up to reversing the order, the gaps are therefore

$$4, 6, 3, 3.$$

The final three gaps sum to $6 + 3 + 3 = 12$, so again two equally colored chosen positions differ by the forbidden distance 12.

All possibilities lead to a contradiction. This proves the lemma. $\square$

Theorem 2.2.

$$\boxed{W\left(x^2 + x, 4\right) = 97.}$$

Proof. **Upper bound.** Suppose that [97] has a valid four coloring for $P(x) = x^2 + x$. Consider the 17 integers

$$1, 7, 13, \ldots, 97.$$

These are exactly the numbers $1 + 6j$ for $0 \le j \le 16$. Give the index j the color of $1 + 6j$. By Lemma 2.1, there are indices $i < j$ with the same color and

$$j - i \in F.$$

Therefore the corresponding integers in [97] have the same color and differ by

$$6(j - i) \in 6F = \{6, 12, 30, 42, 72, 90\}.$$

Each of these is a value of $d^2 + d$:

$$6 = 2^2 + 2, \quad 12 = 3^2 + 3, \quad 30 = 5^2 + 5,$$
$$42 = 6^2 + 6, \quad 72 = 8^2 + 8, \quad 90 = 9^2 + 9.$$

Thus the assumed coloring contains an equally colored pair at a forbidden polynomial distance, a contradiction. Hence

$$W\left(x^2 + x, 4\right) \le 97.$$

Lower bound. The nonzero distances in $\mathcal{D}(x^2 + x)$ that are smaller than 97 are

$$2, 6, 12, 20, 30, 42, 56, 72, 90.$$

The following coloring of [96] was found using the simulated annealing procedure in Algorithm 1. For each of the nine forbidden distances listed above, comparing every pair of entries at that separation shows that the two entries have different colors.

RRGGRRYYRRYYGGBBGGRRGGRRBBRRBBGGYYGGRRBBRRBBRRBB
GGYYGGYYGGYYBBRRBBGGYYGGYYBBYYBBGGBBRRYYRRYYBBYY

Hence a valid four coloring exists on [96], so

$$W\left(x^2 + x, 4\right) \ge 97.$$ □

Theorem 2.3.

$$\boxed{W\left(2x^2 + x, 4\right) = 49.}$$

Proof. **Upper bound.** Apply Lemma 2.1 to the colors of $1 + 3j$ for $0 \le j \le 16$, as in the preceding proof. An equally colored pair has difference in

$$3F = \{3, 6, 15, 21, 36, 45\}.$$

For $P(x) = 2x^2 + x$, these are the values at $d = 1, -2, -3, 3, 4, -5$, respectively. They are all forbidden, so $W\left(2x^2 + x, 4\right) \le 49$.

Lower bound. The values

$$P(m) = \frac{(2m)(2m+1)}{2}, \qquad P(-m) = \frac{(2m-1)(2m)}{2} \qquad (m \ge 1)$$

give every positive triangular number. The forbidden distances below 49 are $1, 3, 6, 10, 15, 21, 28, 36, 45$. The following coloring of [48] was found using Algorithm 1. Direct comparison at each listed distance shows that no forbidden pair has the same color.

RGRYRYGBGRGRBRBGYGRBRBRBGYGYGYBRBGYGYBYBGBRYRYBY

Thus $W\left(2x^2 + x, 4\right) \ge 49$. □

Theorem 2.4.

$$\boxed{W\left(3x^2 + x, 4\right) = 33.}$$

Proof. **Upper bound.** This time apply Lemma 2.1 to the colors of $1 + 2j$ for $0 \le j \le 16$. The resulting difference belongs to

$$2F = \{2, 4, 10, 14, 24, 30\}.$$

For $P(x) = 3x^2 + x$, these values occur at $d = -1, 1, -2, 2, -3, 3$. Hence $W\left(3x^2 + x, 4\right) \le 33$.

Lower bound. The same six values are all the forbidden distances below 33, since $|3d^2 + d| \geq 3d^2 - |d| \geq 44$ for $|d| \geq 4$. The following coloring of [32] was found using Algorithm 1. Entries at each of the six listed distances have different colors.

$$\boxed{\text{GGBBRRGGYYRRGGYYRRBBYYRRBBYYGGBB}}$$

Therefore $W(3x^2 + x, 4) \geq 33$. $\square$

2.1 Odd cycles

Join two integers when their distance is forbidden. The next lemma is used in both upper proofs in this subsection.

Lemma 2.5. *In a proper four coloring, two vertices adjacent to every vertex of the same odd cycle have the same color.*

Proof. The cycle needs at least three colors. Each additional vertex excludes its color from the cycle, so both must use the remaining fourth color. $\square$

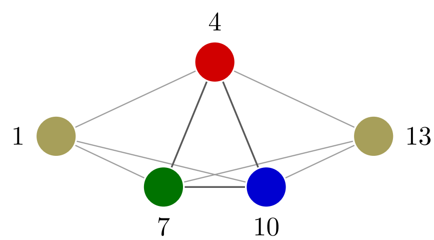

Figure 1: A common triangle for $2x^2 + 7x$. Every drawn edge has forbidden length 3, 6, or 9. The vertices $4, 7, 10$ use three colors, so 1 and 13 have the same fourth color.

Theorem 2.6.

$$\boxed{W\left(2x^2 + 7x, 4\right) = 25.}$$

Proof. **Upper bound.** The forbidden distances below 25 are $3, 4, 5, 6, 9, 15, 22$. Suppose c is a valid four coloring of [25]. For $1 \leq i \leq 13$, the vertices $i+3, i+6, i+9$ form a triangle. Both i and $i+12$ are adjacent to all three. Lemma 2.5 gives

$$c(i) = c(i+12) \qquad (1 \leq i \leq 13).$$

Arrange the positions $1, \ldots, 12$ around a circle. Two positions separated by three, four, five, or six steps around this circle have different colors. Indeed, suitable representatives in [24] differ by one of $3, 4, 5, 6$. Thus each color appears at most three times, and three positions of one color must be consecutive around the circle.

The color of position 1 cannot appear three times. The possible triples would be $(11, 12, 1)$, $(12, 1, 2)$, and $(1, 2, 3)$. They are ruled out, respectively, by the forbidden pairs $(1, 23)$, $(2, 24)$, and $(3, 25)$, each at distance 22, together with the equalities already proved. Hence the four colors cover at most $2 + 3 + 3 + 3 = 11$ of the twelve positions, a contradiction.

Lower bound. The following coloring of [24] was recovered by simulated annealing. Direct comparison at the seven listed distances gives no monochromatic forbidden pair.

$$\boxed{\text{YYYRRRGGGBBBYYYRRRGGGBBB}}$$

Thus $W(2x^2+7x,4) \geq 25$. $\square$

Theorem 2.7.

$$\boxed{W\left(2x^2+9x,4\right)=29.}$$

Proof. **Upper bound.** The forbidden distances below 29 are $4,5,7,9,10,11,18,26$. Suppose c is a valid four coloring of $[29]$. For $5 \leq i \leq 11$, the vertices

$$i-4, \quad i+5, \quad i+9, \quad i+18, \quad i+7$$

form a five cycle in the displayed order. Its edge lengths are $9,4,9,11,11$. Both i and $i+14$ are adjacent to every vertex of this cycle, so Lemma 2.5 gives

$$c(i)=c(i+14) \qquad (5 \leq i \leq 11).$$

Now let $1 \leq i \leq 4$ and consider the colors at

$$i+4, \quad i+7, \quad i+10, \quad i+5, \quad i+9.$$

The first two consecutive pairs have different colors because $c(i+4)=c(i+18)$ and $c(i+7)=c(i+21)$, while the pairs $(i+7,i+18)$ and $(i+10,i+21)$ have forbidden distance 11. The other three consecutive pairs have forbidden distances $5,4,5$. These inequalities form a five cycle, so at least three colors occur. Both i and $i+14$ are adjacent to all five displayed vertices, so their colors agree. Reflection by $v \mapsto 30-v$ supplies the equalities for $12 \leq i \leq 15$. Thus

$$c(i)=c(i+14) \qquad (1 \leq i \leq 15).$$

Finally, consider the colors of $1,3,6,10,13$. The pairs $(1,6)$, $(1,10)$, $(3,10)$, $(3,13)$, $(6,10)$, and $(6,13)$ have different colors directly from their forbidden distances. The remaining four inequalities follow from the established equalities and these edges:

Colors compared	Forbidden pair	Distance
$c(1), c(3)$	$29, 3$	26
$c(1), c(13)$	$1, 27$	26
$c(3), c(6)$	$17, 6$	11
$c(10), c(13)$	$24, 13$	11

Five distinct colors are required, a contradiction.

Lower bound. The following annealed coloring of [28] has no monochromatic pair at any of the eight listed distances.

$$\boxed{\text{YYYRRRRBBBGGGGYYYYRRRBBBBGGG}}$$

Thus $W(2x^2+9x,4) \geq 29$. $\square$

Proposition 2.8. *For every odd integer b and every $k \geq 2$,*

$$W\Big(x^2 + bx, k\Big) = 2W\Big(2x^2 + bx, k\Big) - 1.$$

Proof. Set $P(x) = x^2 + bx$ and $Q(x) = 2x^2 + bx$. For even $d = 2t$, $P(d) = 2Q(t)$. For odd d, the integer $t = -(d+b)/2$ also satisfies $P(d) = 2Q(t)$. Conversely, $2Q(t) = P(2t)$. Thus $\mathcal{D}(P) = 2\mathcal{D}(Q)$, and Proposition 1.2 gives the result. □

In particular, Theorems 2.6 and 2.7 give

$$\boxed{W\Big(x^2 + 7x, 4\Big) = 49, \qquad W\Big(x^2 + 9x, 4\Big) = 57.}$$

Their lower witnesses are obtained by replacing each letter of the corresponding witness by two copies.

2.2 Block colorings

The next two lower witnesses use repeated blocks. Section 3 gives the general construction.

Proposition 2.9.

$$\boxed{W\Big(3x^2 + 7x, 4\Big) = 41, \qquad W\Big(3x^2 + 11x, 4\Big) = 17.}$$

Proof. **The value** 41**.** Restrict a coloring of $[41]$ to its 21 odd integers, written as $1 + 2j$ for $0 \leq j \leq 20$. The distances $1, 2, 3, 20$ between the corresponding values of j are forbidden, since $2, 4, 6, 40 \in \mathcal{D}(3x^2 + 7x)$. Six values of j of one color would have five consecutive gaps of at least 4. Their total is at most 20, so all five gaps would be 4. The first and last would differ by the forbidden distance 20. Thus each color covers at most five of the 21 positions, a contradiction.

For the lower bound, repeat the four blocks **RR**, **GG**, **BB**, **YY** five times:

RRGGBBYYRRGGBBYYRRGGBBYYRRGGBBYYRRGGBBYY

The forbidden distances below 41 are $2, 4, 6, 10, 20, 26, 40$. An equally colored pair in this word has distance 1 or a distance in $[7, 9]$, $[15, 17]$, $[23, 25]$, or $[31, 33]$. None is forbidden, so the coloring of $[40]$ is valid.

The value 17**.** Restrict $[17]$ to its nine odd integers. After dividing their differences by two, the forbidden distances are $2, 3, 4, 5, 7$. Three positions of one color would have two consecutive gaps from $\{1, 6, 8\}$ with sum at most 8. The only possibilities are $(1, 1)$, $(1, 6)$, and $(6, 1)$. Their sums 2 and 7 are forbidden. Each color therefore covers at most two of the nine positions, again a contradiction.

The smallest positive distance in $\mathcal{D}(3x^2 + 11x)$ is 4. Four constant blocks of length 4 give a valid coloring of $[16]$:

RRRRGGGGBBBBYYYY

Every equally colored pair has distance at most 3. Figure 2 shows all the forbidden pairs. □

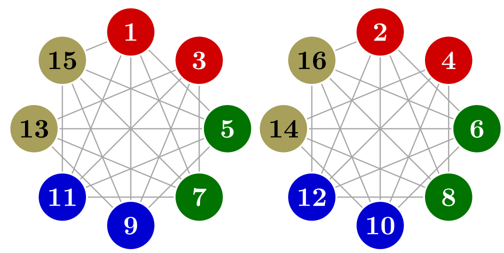

Figure 2: A valid coloring of [16] for $3x^2 + 11x$. The 38 edges have lengths $4, 6, 8, 10, 14$, and every edge has differently colored endpoints. All forbidden distances are even, so the odd and even vertices form separate components.

3 Lower Bounds

The following constructions apply to the forbidden distance set of any nonzero polynomial with zero constant term. They can be combined by taking the largest resulting bound.

Proposition 3.1. *Let $s_1 < s_2$ be the two smallest members of $\mathcal{D}(P)$. Then*

$$\boxed{W(P(x), 4) \geq \max\{4s_1, 2s_2\} + 1.}$$

Proof. Color four consecutive blocks of length s_1 with four different colors. An equally colored pair lies in one block and has distance less than s_1. This gives a valid coloring of $[4s_1]$.

For the other bound, split $[2s_2]$ into two intervals of length s_2. Within either interval, the only possible forbidden distance is s_1. These edges form disjoint paths. Color each path alternately with two colors. Use a different pair of colors on the second interval. □

Proposition 3.2. *Suppose $h, J \geq 1$ are integers, no forbidden distance is smaller than h, and, for $1 \leq j < J$, no forbidden distance lies in*

$$[(4j - 1)h + 1, (4j + 1)h - 1].$$

Then

$$W(P(x), 4) \geq 4hJ + 1.$$

Proof. Repeat J times the word consisting of h copies of each of **R**, **G**, **B**, **Y**, in that order. An equally colored pair in the same block has distance at most $h - 1$. If its blocks are $4j$ blocks apart, its distance lies in the displayed interval. Both possibilities are excluded by the hypotheses. □

The choices $(h, J) = (3, 2), (2, 5), (4, 1)$ give the exact lower bounds for $2x^2 + 7x$, $3x^2 + 7x$, and $3x^2 + 11x$, respectively. These are the block witnesses in Theorem 2.6 and Proposition 2.9. Algorithm 2 checks the conditions for a prescribed interval limit and returns the longest coloring found in this family.

3.1 Lower bounds in terms of the coefficients

We use the following consequence of a theorem of Barajas and Serra [1, Theorem 5].

Lemma 3.3. *Let $s_1 < s_2 < s_3 < s_4$ be positive integers with $s_3 > s_1 + s_2$. The integers have a four coloring avoiding all four distances.*

Proof. After division by their greatest common divisor, every set of four distances requiring five colors has its third smallest distance equal to the sum of the first two [1, Theorem 5]. The strict inequality excludes this case. A coloring for the normalized distances extends along each residue class of the common divisor. □

Theorem 3.4. *For $A, B > 0$, the following lower bounds hold:*

$$\begin{aligned} W(Ax^2+Bx,4) &\geq 9A-3B+1, && 0<B<A,\\ W(Ax^2+Bx,4) &\geq 4A+2B+1, && A<B<2A,\\ W(Ax^2+Bx,4) &\geq \left\lceil \frac{4(B-A)}{3} \right\rceil + 1, && B \geq A. \end{aligned}$$

Proof. If $0 < B < A$, the first five distinct forbidden distances are

$$A-B,\ A+B,\ 4A-2B,\ 4A+2B,\ 9A-3B.$$

The third exceeds the sum of the first two. Lemma 3.3 therefore supplies a four coloring avoiding the first four. Restrict it to $[9A-3B]$, where no other forbidden distance can occur.

If $A < B < 2A$, write $e = B - A$, so $0 < e < A$. Below $4A + 2B = 6A + 2e$, the only possible forbidden distances are

$$e, \quad 2A-2e, \quad 2A+e, \quad 6A-3e.$$

Indeed, the next positive input value is $6A+2e$, and the next negative input value is $12A-4e > 6A+2e$. The first two displayed quantities are the smallest. If they coincide, there are at most three distinct distances, and greedy coloring uses four colors. Otherwise, the third exceeds their sum. Apply Lemma 3.3 and restrict to $[4A + 2B]$.

For the final bound, write $B = A(n + r)$ with $n \geq 1$ an integer and $0 \leq r < 1$. All positive forbidden distances except possibly

$$Anr \quad \text{and} \quad A(n+1)(1-r)$$

are at least $B - A$. To check the negative inputs $-m$ with $1 \leq m < n$, the function $m(B - Am)$ is concave. Its endpoint values are $B - A$ and $A(n - 1)(1 + r)$, both at least $B - A$. When $n = 1$ this range is empty. Inputs with $m \geq n + 2$ give larger values, while positive inputs give at least $A + B$.

The two exceptional quantities have sum $A(n + 1 - r) \geq B - A$. Hence either the smallest positive forbidden distance is at least $(B - A)/3$, or the second smallest is at least $2(B - A)/3$. Zero or repeated exceptional values only remove possible short distances. Proposition 3.1 and integrality give the result. □

Together with Proposition 3.1, these bounds give a lower bound for every nonzero quadratic. On the two special lines, the known exact values and multiplication give $W(Ax^2, 4) = 57A + 1$ and $W(A(x^2 + x), 4) = 96A + 1$. For a nonzero linear polynomial, $W(Bx, 4) = 4B + 1$.

3.2 Adding the fourth color

A periodic three coloring can sometimes be extended beyond its first monochromatic polynomial distance by using a fourth color at selected endpoints.

Lemma 3.5. *Suppose a periodic three coloring of $\mathbb{Z}$ has no monochromatic pair at any polynomial distance not divisible by its period. Let S be the smallest positive forbidden distance divisible by that period. Suppose $1 \leq L \leq S$, $[L]$ has a valid two coloring, and*

$$\mathcal{D}(P) \cap [S-L+1, S+L-1] = \{S\}.$$

Then $W(P(x), 4) \geq S + L + 1$.

Proof. Restrict the periodic coloring to $[S+L]$. Its only monochromatic forbidden pairs are $(i, i+S)$ for $1 \le i \le L$. Choose a valid two coloring of $[L]$. For its first color, recolor i with the fourth color. For its second color, recolor $i+S$ with the fourth color.

Every formerly monochromatic pair has exactly one recolored endpoint. Recolored vertices in either end interval form a valid color class. A distance between the two end intervals lies in $[S-L+1, S+L-1]$. The only forbidden possibility is S, but matching endpoints were assigned opposite choices. Thus the new coloring is valid. $\square$

Proposition 3.6. *Lemma 3.5 gives*

$$W\Big(x^2+x,4\Big) \ge 85, \qquad W\Big(2x^2+x,4\Big) \ge 43.$$

Proof. For $Q(x) = 2x^2 + x$, repeat the period nine word

$$\boxed{\mathbf{RBRGRGBGB}}$$

The residues of $Q(d)$ modulo 9 lie in $\{0, 1, 3, 6\}$. The displayed coloring avoids distances congruent to ± 1 or ± 3, so only distances divisible by 9 can be monochromatic. The smallest such triangular number is $S = 36$. Take $L = 6$. The only forbidden distances on $[6]$ are $1, 3$, so alternating two colors is valid. The interval $[31, 41]$ contains only the triangular number 36. Lemma 3.5 proves the bound 43. Proposition 2.8 then gives the bound 85. $\square$

The period nine construction is the compressed form of the periodic three color witness in Gasarch et al. [3, Theorem 13], where $W\left(x^2+x,3\right) = 73$ is proved.

4 Upper Bound

For $P(x) = Ax^2 + Bx$ with $A, B \ge 0$, label a lattice point (u, v) by $\phi(u, v) = 1 + Bu + Av$. Then

$$\phi(u+d, v+d^2) - \phi(u, v) = Ad^2 + Bd = P(d),$$
$$\phi(u+d, v-d^2) - \phi(u, v) = Bd - Ad^2 = -P(-d).$$

Thus pairs separated by $(d, \pm d^2)$ give polynomial differences. The following finite graph is used for every choice of the coefficients.

Lemma 4.1. *Let $Q = \{0, 1, \ldots, 10\} \times \{0, 1, \ldots, 64\}$. For $1 \le d \le 8$, join (u, v) to $(u+d, v+d^2)$ and to $(u+d, v-d^2)$ whenever both points lie in Q. Every four coloring of Q contains two equally colored joined points.*

Proof. The graph has 715 vertices and 4864 edges. Algorithm 3 exhausts its four colorings and finds no proper coloring. The computation is described in Appendix A. $\square$

An edge of displacement $(d, -d^2)$ has label difference $d(B - Ad)$. If $B = dA$, its two endpoints receive the same integer label, so this edge cannot give a nonzero polynomial difference. Since $1 \le d \le 8$, the only excluded cases are $B = A, 2A, \ldots, 8A$.

Lemma 4.2. *For every $1 \le r \le 8$, $W\left(x^2+rx,4\right) \le 97$.*

Proof. The case $r = 1$ is Theorem 2.2. For $r = 2, \ldots, 8$, Algorithm 3 is applied to $[97]$, joining two integers when their distance is in $\mathcal{D}(x^2+rx)$. In each of the seven cases the search finds no proper four coloring. $\square$

Theorem 4.3. *For all integers a, b, not both zero,*

$$\boxed{W\Big(ax^2 + bx, 4\Big) \leq 64(|a| + |b|) + 1.}$$

Proof. Set $A = |a|$ and $B = |b|$. Replacing $P(x)$ by $-P(x)$ or $P(-x)$ does not change its forbidden distances, so assume $P(x) = Ax^2 + Bx$. If $A = 0$, the identity $W(Bx, 4) = 4B + 1$ from Gasarch et al. [3] gives the result. Hence assume $A > 0$.

First suppose $B \notin \{A, 2A, \ldots, 8A\}$. A valid coloring of $[64A + 10B + 1]$ would color every $(u, v) \in Q$ by the color of $1 + Bu + Av$. Lemma 4.1 gives equally colored points separated by (d, d^2) or $(d, -d^2)$ for some $1 \leq d \leq 8$. Their integer labels differ by $P(d)$ or $-P(-d)$, respectively. The first is positive and the second is nonzero because $B \neq dA$. Either gives a forbidden equally colored pair, a contradiction. Thus

$$W\Big(Ax^2 + Bx, 4\Big) \leq 64A + 10B + 1 \leq 64(A + B) + 1.$$

If $B = rA$ for $1 \leq r \leq 8$, Lemma 4.2 and Proposition 1.2 give

$$W\Big(A(x^2 + rx), 4\Big) \leq 96A + 1 \leq 64(A + B) + 1,$$

since $r \geq 1$. This completes the proof. □

The constant 64 is not claimed to be optimal. The known value $W(x^2, 4) = 58$ [4] and Proposition 1.2 give $W(Ax^2, 4) = 57A + 1$. Consequently, if C_4 is the smallest constant such that $W(ax^2 + bx, 4) \leq C_4(|a| + |b|) + 1$ for all nonzero pairs, then

$$57 \leq C_4 \leq 64.$$

A Computational Methods

The five witnesses found by simulated annealing appear in Section 2. The two witnesses in Proposition 2.9 are block constructions. Exact backtracking proves the finite noncolorability statements in Lemmas 4.1 and 4.2. The general lower bounds in Section 3 are mathematical constructions.

A.1 Simulated Annealing

For a polynomial $P(x)$ and interval $[N]$, let $E_P(N)$ be the set of forbidden pairs. For a four coloring $c : [N] \to \{1, 2, 3, 4\}$, define

$$E(c) = \#\{\{u, v\} \in E_P(N) : c(u) = c(v)\}.$$

Thus $E(c)$ is the number of monochromatic forbidden pairs, and $E(c) = 0$ exactly when c is a valid four coloring.

We search for a coloring with $E(c) = 0$ using simulated annealing. At each step one integer is assigned a different color. If this changes the energy by Δ, the recoloring is always accepted when $\Delta \leq 0$, and is otherwise accepted with probability $e^{-\Delta/T}$. The temperature T decreases during each run according to a geometric cooling schedule. We use the Metropolis acceptance rule [5].

The implementation used to obtain the original three witnesses is `simulatedannealer.cpp`. The resulting colorings are written explicitly in Section 2, so the annealer is used only to find the witnesses.

Algorithm 1 Simulated annealing for the lower bounds

Input: Forbidden pairs $E_P(N)$, restarts R, proposals L, temperatures $T_0, T_{\min}$

Set $\alpha \leftarrow (T_{\min}/T_0)^{1/L}$
for $r = 1, \ldots, R$ **do**
  Choose a random four coloring c and set $T \leftarrow T_0$
  if $E(c) = 0$ **then**
    return c
  end if
  for $s = 1, \ldots, L$ **do**
    Choose a vertex v and a color $q \neq c(v)$ uniformly at random
    Let Δ be the energy change from recoloring v by q, and draw U uniformly from $[0, 1]$
    if $\Delta \leq 0$ or $U < e^{-\Delta/T}$ **then**
      Set $c(v) \leftarrow q$
    end if
    if $E(c) = 0$ **then**
      return c
    end if
    Set $T \leftarrow \alpha T$
  end for
end for
return FAILURE

The added instances $(a, b, N) = (2, 7, 24)$ and $(2, 9, 28)$ were also run with a Python implementation of the same rule. The runs used $T_0 = 2$, $T_{\min} = 0.005$, and $L = 100000$ proposals per restart, with seeds 20261111 and 20261113, respectively. Both displayed witnesses were obtained in the first restart and checked against every forbidden pair.

Randomized local search has previously been used for polynomial van der Waerden computations [4]. WalkSAT provides a related stochastic local search approach to satisfying the coloring constraints.

A.2 Block search

Algorithm 2 applies Proposition 3.2 with an interval limit M. The input set contains every forbidden distance below M. For each admissible block length, the algorithm adds complete four block repetitions until the next distance interval meets the input set or the length limit is reached. It returns a valid witness together with its lower bound. The coefficient and fourth color extension bounds in Section 3 are applied directly from their proofs.

Algorithm 2 Lower bounds from repeated blocks

Input: $M \geq 4$, $D = \mathcal{D}(P) \cap \{1, \ldots, M-1\}$

Set $L \leftarrow 0$
for $h = 1, \ldots, \lfloor M/4 \rfloor$ **do**
  if $D \cap \{1, \ldots, h-1\} = \varnothing$ **then**
    Set $J \leftarrow 1$
    while $4h(J+1) \leq M$ and $D \cap [(4J-1)h+1, (4J+1)h-1] = \varnothing$ **do**
      Set $J \leftarrow J+1$
    end while
    if $4hJ > L$ **then**
      Set $L \leftarrow 4hJ$
      Let c be J repetitions of h copies of each color **R**, **G**, **B**, **Y**
    end if
  end if
end for
return the bound $L+1$ and coloring c of $[L]$

A.3 Exact search

The remaining computations are ordinary finite four coloring problems. The same exact search is used for both Lemma 4.1 and Lemma 4.2.

The search uses the saturation degree rule of Brélaz [6]. At each step it chooses an uncolored vertex having the largest number of different colors among its colored neighbors. It then tries every possible color. Colors not yet used are equivalent under permutation, so only one new color needs to be tried. A branch is discarded as soon as an uncolored vertex has neighbors in all four colors.

Algorithm 3 Exact four color search

Input: Finite vertex set V and forbidden pair set E

function SEARCH(c)
  if every vertex is colored **then**
    return TRUE
  end if
  Choose an uncolored vertex v of maximum saturation degree
  Break ties by the number of uncolored neighbors
  Let T be the colors not used on colored neighbors of v
  Keep every color in T already used in the current partial coloring, but at most one color not yet used
  for each $q \in T$ **do**
    Give v color q
    if no uncolored vertex sees all four colors **then**
      if SEARCH(c) **then**
        return TRUE
      end if
    end if
    Uncolor v
  end for
  return FALSE
end function
return SEARCH(empty coloring)

For Lemma 4.1, the input consists of the 715 points of Q and its 4864 forbidden pairs. For Lemma 4.2, the same search is run on [97] for each $P(x) = x^2 + rx$, $2 \le r \le 8$. In every case used in the paper, the search returns FALSE.

The implementation is `fourcolorsearch.cpp`.

A.4 Comparison of bounds

Table 1 lists the lower bound, the exact value, and the upper bound for each polynomial. The lower bounds follow from the constructions in Section 3 and Proposition 2.8.

Polynomial	LB	Exact	UB
$x^2 + x$	85	97	129
$2x^2 + x$	43	49	193
$3x^2 + x$	25	33	257
$2x^2 + 7x$	25	25	577
$x^2 + 7x$	49	49	513
$2x^2 + 9x$	17	29	705
$x^2 + 9x$	33	57	641
$3x^2 + 7x$	41	41	641
$3x^2 + 11x$	17	17	897

Table 1: Comparing the lower and upper bounds with the known exact values

The lower construction is sharp for $2x^2 + 7x$, $3x^2 + 7x$, $3x^2 + 11x$, and the transferred example $x^2 + 7x$. The universal upper bound exceeds all nine exact values.

References

[1] J. Barajas and O. Serra, Distance graphs with maximum chromatic number, *Discrete Mathematics and Theoretical Computer Science Proceedings*, AE (2005), 171–174. `https://dmtcs.episciences.org/3391`.

[2] V. Bergelson and A. Leibman, Polynomial extensions of van der Waerden's and Szemerédi's theorems, *J. Amer. Math. Soc.* **9** (1996), 725–753.

[3] W. Gasarch, C. Kruskal, J. Kruskal, and Z. Price, High school proofs for better bounds on the quadratic van der Waerden numbers, available at `https://www.cs.umd.edu/~gasarch/GKKZP/gkkp.pdf`.

[4] B. Canakci, H. Christenson, R. Fleischman, W. Gasarch, N. McNabb, and D. Smolyak, On SAT solvers and Ramsey-type numbers, arXiv:2312.01159, 2023.

[5] S. Kirkpatrick, C. D. Gelatt, Jr., and M. P. Vecchi, Optimization by simulated annealing, *Science* **220** (1983), 671–680.

[6] D. Brélaz, New methods to color the vertices of a graph, *Commun. ACM* **22** (1979), 251–256.